Partial Solutions of Exercises found in College Algebra Notes

Subject to change.

Contents

1 Some Arithmetic and Geometry of Real Numbers

Exercise 1.17.

(a) Explain with cases why, if a is a real number, then $a \leq |a|$.

(b) Explain, if, for any real number a, $\text{sgn}(a) = 1$ if $a > 0$, $\text{sgn}(a) = -1$ if $a < 0$ and $\text{sgn}(a) = 0$ if $a = 0$, then $a = \text{sgn}(a)|a|$ and $|a| = \text{sgn}(a)a$.

Solution:

(a) Suppose a is a real number. Then either $a \geq 0$ or $a < 0$ by Trichotomy of order.

- If $a \geq 0$, then $|a| = a \geq a$.
- If $a < 0$, then $|a| = -a > 0 > a$.

In any case, $|a| \geq a$.

(b) Suppose a is a real number. Then either $a > 0$, $a = 0$ or $a < 0$ by Trichotomy of order.

- If $a > 0$, then $|a| = a$ and $\text{sgn}(a) = 1$. Hence, $a = 1 \cdot a = \text{sgn}(a)|a|$. Also, $|a| = a = 1 \cdot a = \text{sgn}(a)a$.
- If $a = 0$, then $|a| = 0$ and $\text{sgn}(a) = 0$. Hence, $a = 0 \cdot 0 = \text{sgn}(a)|a|$. Also, $|a| = 0 = 0 \cdot 0 = \text{sgn}(a)a$.
- If $a < 0$, then $|a| = -a$ and $\text{sgn}(a) = -1$. Hence, $a = (-1) \cdot (-a) = \text{sgn}(a)|a|$. Also, $|a| = -a = (-1) \cdot a = \text{sgn}(a)a$. The claims $a = (-1) \cdot (-a)$ and $-a = (-1) \cdot a$ for any real number a are not definitions, but can be explained using the distributive property.

In any case, $a = \text{sgn}(a)|a|$ and $|a| = \text{sgn}(a)a$.

2 Relations and Functions

Exercise 2.9. In Example **??**, determine which are functions as relations in $\mathbb{R}^2$ and explain why.

Solution:

(a) $<$ is the set $\{(a,b) \in \mathbb{R}^2 \mid a < b\}$. This is not a function. For example, both $0 < 1$ and $0 < 2$. Importantly, $1 \neq 2$. That is, $(0,1) \in <$ and $(0,2) \in <$ (there are two outputs for the same input). Another way to phrase this is $<$ fails the vertical line test.

(b) $=$ on $\mathbb{R}^2$ is the set $\{(a,b) \in \mathbb{R}^2 \mid a = b\}$. This is a function. For any $a \in \mathbb{R}$, if $a = b$ and $a = c$ for two real numbers b and c, then $b = c$ by transitivity of equality.

(c) The horizontal axis is the set $\{(a,b) \in \mathbb{R}^2 \mid b = 0\}$. This is a function. For every $a \in \mathbb{R}$, if (a,b) and (a,c) are on the horizontal axis for some real numbers b and c, then $b = 0 = c$, by definition.

(d) The vertical axis is the set $\{(a,b) \in \mathbb{R}^2 \mid a = 0\}$. This is not a function. For example, both $(0,1)$ and $(0,2)$ are members of this set. So, there exists an input which has two outputs.

Exercise 2.10.

For this exercise, let $B = \{a, b\}$, the set consisting of the characters a and b.

(a) If $A_1 = \{1\}$, the set consisting of the number 1, describe all functions with domain A_1 and codomain B.

(b) If $A_2 = \{1, 2\}$, describe all functions with domain A_2 and codomain B.

(c) If $A_3 = \{1, 2, 3\}$, describe all functions with domain A_3 and codomain B.

(d) If n is a natural number (an integer > 0), try to deduce a formula for the number of functions with domain consisting of n elements and codomain B. Can you explain your result?

Solution: Remember what it means to be a function. **Each** input is assigned **exactly one** output.

(a) There are two functions.

A_1	B
1	a

,

A_1	B
1	b

.

(b) There are four functions.

A_2	B
1	a
2	a

,

A_2	B
1	a
2	b

,

A_2	B
1	b
2	a

,

A_2	B
1	b
2	b

.

(c) There are eight functions.

A_3	B
1	a
2	a
3	a

,

A_3	B
1	a
2	a
3	b

,

A_3	B
1	a
2	b
3	a

,

A_3	B
1	a
2	b
3	b

,

A_3	B
1	b
2	a
3	a

,

A_3	B
1	b
2	a
3	b

,

A_3	B
1	b
2	b
3	a

,

A_3	B
1	b
2	b
3	b

.

(d) Notice each time we double the amount of functions. This is because there are two elements in the set B. If n is a natural number greater than 1, for every function f between A_{n-1} and B, there are exactly two functions between A_n and B whose restriction to A_{n-1}, as a subset of A_n, is f. One of these functions sends n to a, the other sends n to b. So, the number of functions with domain A_n and target B is 2^n.

Exercise 2.17.

(a) Express the real number line as a disjoint union of two intervals.

(b) Express the real number line as a disjoint union of three intervals.

(c) Express the relation $\leq$ as a union of two relations.

Solution:

(a) There are infinitely many intervals to choose from. We choose $(-\infty, \infty) = (-\infty, 0) \cup [0, \infty)$. This equality (and the disjointness of the intervals on the right) is the statement of Trichotomy of positivity. Any number is exactly either negative, zero, or positive.

(b) $(-\infty, \infty) = (-\infty, 0) \cup [0, 1] \cup (1, \infty)$. Let's demonstrate these two sets are equal. Certainly, $(-\infty, 0) \cup [0, 1] \cup (1, \infty)$ is a subset of $(-\infty, \infty)$. If $x \in (-\infty, \infty)$, then either $x < 0$, meaning $x \in (-\infty, 0)$, $x \geq 0$ and $x \leq 1$, meaning $x \in [0, 1]$, or $x > 1$, meaning $x \in (1, \infty)$. These follow from repeated use of Trichotomy of order. Hence, $x \in (-\infty, 0) \cup [0, 1] \cup (1, \infty)$. Since x is an arbitrary real number, it follows $\mathbb{R} = (-\infty, 0) \cup [0, 1] \cup (1, \infty)$.

Moreover, the intervals $(-\infty, 0)$, $[0, 1]$ and $(1, \infty)$ are mutually disjoint by Trichotomy of order. If $x < 0$, then it's not true that $x \geq 0$. So, if $x \in (0, \infty)$, then $x \notin [0, 1] \cup (1, \infty)$. Similarly, if $0 \leq x \leq 1$, then it is not true that $x < 0$ or $x > 1$. So, if $x \in [0, 1]$, then $x \notin (-\infty, 0) \cup (1, \infty)$. Finally, if $x > 1$, then it is not true that $x \leq 1$. Hence, if $x \in (1, \infty)$, then $x \notin (-\infty, 0) \cup [0, 1]$.

(c) The relation $\leq$ is the relation $< \cup =$. If $a \leq b$, then either $a < b$ or $a = b$ by definition. Hence, $\leq$ is a subset of $< \cup =$. If either $a < b$ or $a = b$, then $a \leq b$ by definition. Hence, $< \cup =$ is a subset of $\leq$. Since these relations are subsets of each other, they are equal.

Exercise 2.20. Determine when the union of two functions $f \cup g$ is itself a function. Such a function is called a **piecewise function**. Hint: If the domain of f and the domain of g are disjoint, then $f \cup g$ is a function.

Solution: Suppose f and g are functions. By definition, the relation $f \cup g$ is a function exactly when, whenever $(x, y) \in f \cup g$ and $(x, z) \in f \cup g$, $y = z$. Now by definition, $(x, y) \in f \cup g$ if either $(x, y) \in f$ or $(x, y) \in g$ and $(x, z) \in f \cup g$ if either $(x, z) \in f$ or $(x, z) \in g$. So, if (x, y) and (x, z) are members of f or both members of g, then $y = z$ because f and g are functions themselves. So, we need only consider when (x, y) is a member of f and (x, z) is a member of g or vice versa. In this case, $x \in \text{domain}(f) \cap \text{domain}(g)$, the intersection of the domains of f and g. In order to ensure $y = z$, f must agree with g on the intersection of their domain.

Claim: $f \cup g$ is a function if and only if, whenever $x \in \text{domain}(f) \cap \text{domain}(g)$, $f(x) = g(x)$.
Proof:

- Suppose $f \cup g$ is a function. Now fix $x \in \text{domain}(f) \cap \text{domain}(g)$. Then $(x, f(x)) \in f$ and $(x, g(x)) \in g$. Since $f \cup g$ is a function, $f(x) = g(x)$.
- Conversely, suppose, whenever $x \in \text{domain}(f) \cap \text{domain}(g)$, $f(x) = g(x)$. Further, to demonstrate $f \cup g$ is a function under this hypothesis, suppose $(x, y) \in f \cup g$ and $(x, z) \in f \cup g$. Then either (x, y) and (x, z) are members of either f or g separately, in which case $y = z$ since f and g are functions, or $x \in \text{domain}(f) \cap \text{domain}(g)$, in which case, $y = f(x) = g(x) = z$, where $f(x) = g(x)$ by hypothesis. Hence, $f \cup g$ is a function.

In the language of function restriction, this claim can be rephrased as $f \cup g$ is a function if and only if $f|_{\text{domain}(f) \cap \text{domain}(g)} = g|_{\text{domain}(f) \cap \text{domain}(g)}$.

Exercise 2.25. Suppose f and g are both functions.

(n) If A is a subset of the codomain of f, explain why $f(f^{-1}(A))$ is a subset of A.

(o) Give an example of a function f and subset of its codomain A for which A is not a subset of $f(f^{-1}(A))$.

(p) Find the domain of $f \circ g$ in terms of the domain of f and g as a preimage.

Solution:

(n) Suppose A is a subset of the codomain of f. Recall the image of $f^{-1}(A)$ under f is the set

$$f(f^{-1}(A)) = \{y \in \text{codomain}(f) \mid \text{ there is some } x \in f^{-1}(A) \text{ such that } f(x) = y\}.$$

Recall the preimage of A under f is the set

$$f^{-1}(A) = \{x \in \text{domain}(f) \mid f(x) \in A\}.$$

We want to demonstrate the image of the preimage of A under f is a subset of A. If $y \in f(f^{-1}(A))$, then there is some $x \in f^{-1}(A)$ such that $f(x) = y$. Since $x \in f^{-1}(A)$, then $f(x) \in A$ by definition. Since $y = f(x)$, $y \in A$. Since y is an arbitrary element of $f(f^{-1}(A))$, we conclude $f(f^{-1}(A))$ is a subset of A.

(o) Define f with domain $\{0\}$ and codomain $\{1, 2\}$ by $f(0) = 1$. Now, if $A = \{1, 2\}$, the codomain itself, then $f^{-1}(A) = \{0\}$, since $f^{-1}(\text{codomain}(f)) = \text{domain}(f)$ for all functions f. Then $f(f^{-1}(A)) = \{1\}$, since $f(\text{domain}(f)) = \text{range}(f)$ for all functions f. Hence, A is not a subset of $f(f^{-1}(A))$.

As another example, the square function, sq, has domain $\mathbb{R}$ and codomain $\mathbb{R}$, but range $[0, \infty)$. If $A = [-1, 1]$, then $f^{-1}(A) = [-1, 1]$, since if $-1 \leq x^2 \leq 1$, then $-1 \leq x \leq 1$. Then $f(f^{-1}(A)) = [0, 1]$, since if $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$. Hence, A is not a subset of $f(f^{-1}(A))$.

In both cases, A is chosen so that some of its members are not in the range of f. This is the only condition needed to ensure A is not a subset of $f(f^{-1}(A))$.

(p) Suppose f and g are functions. Recall the domain of the composition of f with g, $f \circ g$, is defined to be the set of all $x \in \text{domain}(g)$ such that $g(x) \in \text{domain}(f)$. So,

$$\text{domain}(f \circ g) = g^{-1}(\text{domain}(f)).$$

Exercise 2.26. If f, g, l, m are functions for which $f \cup g$ and $l \cup m$ are functions, find the domain of $(f \cup g) \circ (l \cup m)$ as a union of preimages.

Solution: Suppose f, g, l, m are functions for which $f \cup g$ and $l \cup m$ are functions. By Theorem 2.23 (Composition of Piecewise) in the notes, we know

$$(f \cup g) \circ (l \cup m) = f \circ l \cup \ g \circ l \cup \ f \circ m \cup \ g \circ m.$$

By Definition of the union of functions,

$$\text{domain}((f \cup g) \circ (l \cup m)) = \text{domain}(f \circ l) \cup \ \text{domain}(g \circ l) \cup \ \text{domain}(f \circ m) \cup \ \text{domain}(g \circ m)$$

By Exercise 2.25 p,

$$\text{domain}((f \cup g) \circ (l \cup m)) = l^{-1}(\text{domain}(f)) \cup l^{-1}(\text{domain}(g)) \cup m^{-1}(\text{domain}(f)) \cup m^{-1}(\text{domain}(g)).$$

Exercise 2.36.

(b) The **difference** of two real-valued functions f and g is defined by

$$f - g := f + \text{const}_{-1} \cdot g.$$

That is, for all $x \in \text{domain}(f) \cap \text{domain}(g)$, $f - g(x) = f(x) - g(x)$. Express sgn as the the difference of two indicator functions.

Solution:

(b) Recall, for a given set B, and given subset A of B, the indicator function of A of B is defined to be

$$\chi_{A,B}(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

for all $x \in B$. $\chi_{A,B}(x)$ tells us whether x is in A or x is not in A. $\chi_{A,B}$ distinguishes members of A apart from other members of B.

The sgn function is defined to be

$$\text{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

for all $x \in \mathbb{R}$. $\text{sgn}(x)$ tells us whether x is positive, zero, or negative. By Trichotomy, these sets are disjoint. It's not difficult to verify

$$\text{sgn} = \chi_{[0,\infty),\mathbb{R}} - \chi_{(-\infty,0],\mathbb{R}}.$$

Notice these are both closed unbounded intervals. Notice also

$$\text{sgn} = \chi_{(0,\infty),\mathbb{R}} - \chi_{(-\infty,0),\mathbb{R}}.$$

Exercise 2.38.

(d) If m_1, m_2, b_1, b_2 are real numbers, express $l_{m_2,b_2} \cdot l_{m_1,b_1}$ as a quadratic function.

(f) Demonstrate by way of example. The degree of the sum of two polynomials can be equal to the maximum of their degrees.

(h) Provide an example of two degree three polynomials whose sum has degree 0.

Solution:

(d) For every $x \in \mathbb{R}$,

$$l_{m_2,b_2} \cdot l_{m_1,b_1}(x) = (m_2x + b_2)(m_1x + b_1) = m_1m_2x^2 + (m_1b_2 + m_2b_1)x + b_1b_2.$$

Hence,

$$l_{m_2,b_2} \cdot l_{m_1,b_1} = q_{m_1m_2, m_1b_2+m_2b_1, b_1b_2}.$$

(f) If $f(x) = x$ (degree 1) and $g(x) = 1$ (degree 0), then $(f + g)(x) = x + 1$ (degree 1).

(h) If $f(x) = x^3$ and $g(x) = -x^3+1$, then f and g both have degree 3 and $(f+g)(x) = x^3-x^3+1 = 1$. This means $f + g$ is a nonzero constant and hence has degree 0.

4 Polynomials

Exercise 4.11.

(a) Is there any real number c for which $q_{1,-(c+1),c}$ does not have real zeros?

(b) Find $a \in \mathbb{R}$ for which $q_{1,0,a}$ has

- no zeros,
- one zero, and
- two zeros.

Solution:

(a) The zeros of $q_{1,-(c+1),c}$ are the numbers

$$\left\{ \frac{c+1 \pm \sqrt{(c+1)^2 - 4c}}{2} \right\} = \left\{ \frac{c+1 \pm \sqrt{(c-1)^2}}{2} \right\} = \left\{ \frac{c+1 \pm (c-1)}{2} \right\} = \{c, 1\}\,.$$

These are real numbers if c is a real number.

(b) Note $q_{1,0,a}(x) = x^2 + a$ and $(x^2 + a) = (x - \sqrt{a})(x + \sqrt{a})$ for all real x.

- If $a < 0$, then $q_{1,0,a}$ has no real zeros.
- If $a = 0$, then $q_{1,0,a}$ has one real zero.
- If $a > 0$, then $q_{1,0,a}$ has two real zeros.

Exercise 4.14. Suppose f is any polynomial.

(a) If d is a nonzero constant polynomial, what are the unique polynomials q and r for which $f = q \cdot d + r$ and the degree of r is strictly less than the degree of d?

(b) If the degree of f is equal to the degree of d, what are the unique polynomials q and r for which $f = q \cdot d + r$ and the degree of r is strictly less than the degree of d?

(c) If the degree of f is strictly less than the degree of d, what are the unique polynomials q and r for which $f = q \cdot d + r$ and the degree of r is strictly less than the degree of d?

(d) Construct polynomials f, d, q, r for which $f = q \cdot d + r$ and the degree of r is two less than the degree of d and the degree of q is equal to the degree of d.

Solution:

(a) If d is a nonzero constant polynomial (degree 0), then r must be the zero polynomial, since the zero polynomial is the only polynomial with degree less than 0. Then $q = \frac{f}{d}$. Since f is a polynomial and d is a nonzero constant function, then q is a polynomial.

(b) If the degree of f is equal to the degree of d, then q must be a nonzero constant, since the degree of a product of two polynomials is the sum of their degrees. If a is the leading coefficient of f and b is the leading coefficient of d, then a and b are not zero. Define

$$q = \text{const}_{\frac{a}{b}}$$

and

$$r = f - q \cdot d.$$

Notice the degree and leading coefficient of f and $q \cdot d$ are the same. Hence, their difference has degree less than f, and hence d. So, the degree of r is less than the degree of d. Moreover,

$$f = q \cdot d + f - q \cdot d = q \cdot d + r.$$

(c) If the degree of f is strictly less than the degree of d, define $q = \text{const}_0$ and $r = f$. Then the degree of r is less than the degree of d and $f = \text{const}_0 \cdot d + f = q \cdot d + r$.

(d) If $r = \text{const}_1$, $d = q = \text{sq}$, and $f(x) = x^4 + 1$ for every real x, then the degree of d and q are both two, the degree of r is zero, and $f = q \cdot d + r$.

Exercise 4.16.

(a) Find q in the conclusion of the remainder theorem.

(b) Generalize the remainder theorem to work for $l_{m,d}$ if $m \neq 0$ instead of only $l_{1,-k}$. In other words, what is the remainder after dividing f by $l_{m,d}$ if $m \neq 0$?

Solution:

(a) For any polynomial f, for any real number k, we wish to find polynomial q such that $f = q \cdot l_{1,-k} + \text{const}_{f(k)}$. We can write $f(x) = \sum_{i=0}^{n} a_i(x-k)^i$ for some a_i, n and every x. In this case, $a_0 = f(k)$ and $q(x) = \sum_{i=1}^{n} a_i(x-k)^{i-1}$.

(b) The remainder will still be a constant, since the degree of the divisor is one. Hence, $f = q \cdot l_{m,d} + r$ for some polynomial q and some constant r. Then $f\left(-\frac{d}{m}\right) = q\left(-\frac{d}{m}\right) \cdot \left(m\left(-\frac{d}{m}\right) + d\right) + r = r$. Hence,

$$r = f\left(-\frac{d}{m}\right).$$

Exercise 4.21.

1. Give an example of a real function which is positive on the closed interval $[-1, 0]$ and is negative on the half-open interval $(0, 1]$.

2. Is it possible for this function to be a polynomial? Why or why not?

Solution:

1. Let $f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ -1 & \text{if } 0 < x \leq 1. \end{cases}$

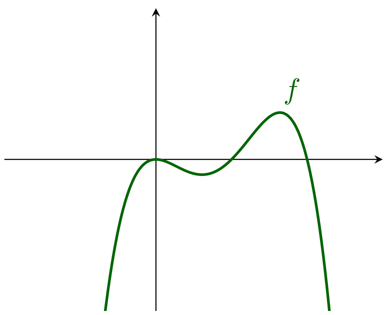
2. No. This is by the intermediate value theorem. If f is a polynomial which is positive on $[-1, 0)$ and negative on $(0, 1]$, then $f(0) = 0$ necessarily by the intermediate value theorem. Hence, $f(0)$ is not positive. So, $f(0)$ cannot be positive if f is negative on $(0, 1]$.

Exercise 4.27.

1. Sketch the graph of a polynomial which has exactly two local maximum values, one local minimum value and three zeros. *Hint:* Think about what the multiplicity of at least one of the zeros has to be.

2. Find an equation for the polynomial found in part (a).

Solution:

1. 

2. $f(x) = -x^2(x-1)(x-2)$ for all x.

5 Rational Functions

Exercise 5.10.

(a) If a is a real number, does the rational function

$$l_{1,-a} \cdot (\text{recip} \circ q_{1,-2a,a^2})$$

have any infinite limits at a point?

(b) If a is a real number, does the rational function

$$q_{1,-2a,a^2} \cdot (\text{recip} \circ l_{1,-a})$$

have any infinite limits at a point?

(c) Does the rational function

$$l_{1,0} \cdot (\text{recip} \circ q_{1,0,1})$$

have any infinite limits at a point?

Solution: Note $q_{1,-2a,a^2} = l_{1,-a} \cdot l_{1,-a}$. Hence, the multiplicity of a as a zero of $l_{1,-a}$ is one and the multiplicity of a as a zero of $q_{1,-2a,a^2}$ is two.

(a) By Theorem 5.6(a), this rational function has an infinite limit at a. In particular,

$$\lim_{a^-} l_{1,-a} \cdot (\text{recip} \circ q_{1,-2a,a^2}) = -\infty$$

and

$$\lim_{a^+} l_{1,-a} \cdot (\text{recip} \circ q_{1,-2a,a^2}) = \infty.$$

(b) By Theorem 5.11, this rational function does not have an infinite limit at a, since it has a removable discontinuity there. In fact,

$$\lim_{a^\pm} q_{1,-2a,a^2} \cdot (\text{recip} \circ l_{1,-a}) = 0.$$

And the domain of this rational function is $\mathbb{R} \setminus \{a\}$. So, this function does not have any infinite limits at a point.

(c) No. This is because $q_{1,0,1}$ does not have any real zeros. This means the domain of $l_{1,0} \cdot (\text{recip} \circ q_{1,0,1})$ is $\mathbb{R}$.

6 Invertibility and Radical Functions

Exercise 6.5.

(c) When is an affine function $l_{m,d}$, for real numbers m, d, invertible? What is its inverse in this case?

(d) Find all the affine functions which are their own inverse.

(e) For which real numbers a is $l_{1,a}$ invertible? For those a, find the inverse of $l_{1,a}$. $l_{1,a}$ is called addition on the right by a.

(f) For which real numbers a is $l_{a,0}$ invertible? For those a, find the inverse of $l_{a,0}$. $l_{a,0}$ is called multiplication on the left by a.

Solution:

(c) Suppose $l_{m,d}$ is invertible. In particular, $l_{m,d}$ is one-to-one. That is, $l_{m,d}(x) = l_{m,d}(y)$ implies $x = y$. But $l_{m,d}(x) = l_{m,d}(y)$ if and only if $mx + d = my + d$ if and only if $mx = my$. And $mx = my$ implies $x = y$ if and only if $m \neq 0$. So, we conclude $l_{m,d}$ is invertible if and only if $m \neq 0$. If $m \neq 0$, then

$$l_{m,d}^{-1} = l_{m^{-1},-m^{-1}d}.$$

To check, for all $x \in \mathbb{R}$,

$$l_{m^{-1},-m^{-1}d}(l_{m,d}(x)) = m^{-1}(mx + d) - m^{-1}d = x$$

and

$$l_{m,d}(l_{m^{-1},-m^{-1}d}(x)) = m(m^{-1}x - m^{-1}d) + d = x.$$

Hence,

$$l_{m^{-1},-m^{-1}d} \circ l_{m,d} = \mathrm{id}_{\mathbb{R}}$$

and

$$l_{m,d} \circ l_{m^{-1},-m^{-1}d} = \mathrm{id}_{\mathbb{R}}.$$

(d) We note $l_{m,d} \circ l_{m,d} = \mathrm{id}_{\mathbb{R}}$ if and only if, for all $x \in \mathbb{R}$,

$$m(mx + d) + d = x$$

if and only if, for all $x \in \mathbb{R}$,

$$(m^2 - 1)x + (m + 1)d = 0.$$

And $m^2 - 1 = (m + 1)(m - 1)$. By Exercise 2.38(a), a polynomial is completely determined by its coefficients. So, if a polynomial is identically zero, then its coefficients are. That is,

$$\begin{cases} (m + 1)(m - 1) = 0 \\ (m + 1)d = 0. \end{cases}$$

Hence, by second equation and the zero product property, either $m + 1 = 0$ or $d = 0$. If $m + 1 = 0$, then $m = -1$. If $d = 0$ but $m + 1 \neq 0$, then by the first equation and the zero product property, $m = 1$. Hence, if $l_{m,d}$ is its own inverse for some real m, d, then either $m = 1$ and $d = 0$, which is $\mathrm{id}_{\mathbb{R}}$, or $m = -1$ and d is arbitrary. That is, the only lines which are their own inverse are the identity function and any line perpendicular to the identity.

(e) For any number $a \in \mathbb{R}$, $l_{1,a}$ is invertible. In this case, the inverse of $l_{1,a}$ is $l_{1,-a}$, since $x+a-a = x$ and $x-a+a = x$ for all real x. In words, the inverse of addition on the right by a is subtraction on the right by a.

(f) For any nonzero number $a \in \mathbb{R}$, $l_{a,0}$ is invertible. In this case, the inverse of $l_{a,0}$ is $l_{a^{-1},0}$, since $aa^{-1}x = x$ and $a^{-1}ax = x$ for all $x \in \mathbb{R}$. In words, if $a \neq 0$, the inverse of multiplication on the left by a is multiplication on the left by a^{-1}. If $a = 0$, $l_{a,0}$ is not one-to-one because it's the constant zero function, hence it is not invertible.

Exercise 6.7. Suppose a is a nonzero real number.

(g) Find the inverse of id^a as another power function.

(i) Explain why $\mathrm{id}^{\frac{1}{2}} \circ \mathrm{sq} = \mathrm{abs} \setminus \{(0,0)\}$ as relations by way of Theorem **??**.

(j) Find the domain, range and inverse of the function

$$R = l_{1,1} \circ \mathrm{id}^3 \cdot (\mathrm{recip} \circ l_{1,-8} \circ \mathrm{id}^3).$$

Solution:

(g) If f is the inverse of id^a, then

$$x = f(\mathrm{id}^a(x)) = f(x^a)$$

for all $x > 0$ and

$$(f(y))^a = \mathrm{id}^a(f(y)) = y$$

for all y in the domain of f. This last equation implies

$$f(y) = y^{\frac{1}{a}}$$

for all y in the domain of f. The first equation implies, if $y = x^a$, then $x = y^{\frac{1}{a}}$ and

$$y^{\frac{1}{a}} = f(y).$$

And recall, for every positive number x, there is a unique positive number y such that $x = y^{\frac{1}{a}}$. Hence, we conclude

$$(\mathrm{id}^a)^{-1} = \mathrm{id}^{\frac{1}{a}}.$$

(i) First, the domain of $\mathrm{id}^{\frac{1}{2}} \circ \mathrm{sq}$ is $\mathbb{R} \setminus \{0\}$ since

$$\mathrm{domain}(\mathrm{id}^{\frac{1}{2}} \circ \mathrm{sq}) = \mathrm{sq}^{-1}(\mathrm{domain}(\mathrm{id}^{\frac{1}{2}})) = \mathrm{sq}^{-1}((0,\infty)) = \mathbb{R} \setminus \{0\}.$$

The last equality is the statement of Theorem 4.2(a). $\mathbb{R} \setminus \{0\}$ is the domain of $\mathrm{abs} \setminus \{(0,0)\}$ as well.

Next, if

$$(x,y) \in \mathrm{id}^{\frac{1}{2}} \circ \mathrm{sq},$$

then $x \neq 0$ and $y = \sqrt{x^2}$. If $x > 0$, then $y = \sqrt{x^2} = x = |x|$. Hence,

$$(x,y) = (x,|x|) \in \mathrm{abs} \setminus \{(0,0)\}.$$

If $x < 0$, then $-x > 0$, and $y = \sqrt{x^2} = \sqrt{(-x)^2} = -x = |x|$. Hence,

$$(x,y) = (x,|x|) \in \mathrm{abs} \setminus \{(0,0)\}$$

in this case as well.

If

$$(x, y) \in \text{abs} \setminus \{(0,0)\},$$

then $x \neq 0$ and $y = |x|$. As we have just demonstrated, $|x| = \sqrt{x^2}$. Hence,

$$(x, y) \in \text{id}^{\frac{1}{2}} \circ \text{sq}\,.$$

As an aside, how do we justify the statement $x > 0$ implies $\sqrt{x^2} = x$? We know, if $b \geq 0$, there is a unique $a \geq 0$ such that $b = a^2$. Namely, $a = \sqrt{b}$. Since $x^2 > 0$, there is a unique $a > 0$ such that $x^2 = a^2$. Since sq is strictly increasing on $(0, \infty)$, it is one-to-one there. Hence, $a = x$. That is, $\sqrt{x^2} = a = x$. Similarly, if $x < 0$, then $-x > 0$, and we can apply the same reasoning to conclude $\sqrt{x^2} = \sqrt{(-x)^2} = -x$. $\sqrt{x^2} = \sqrt{(-x)^2}$ because $x^2 = (-x)^2$ and $\text{id}^{1/2}$ is a function.

(j) The domain of R is

$$\text{domain}(l_{1,1} \circ \text{id}^3) \cap \text{domain}(\text{recip} \circ l_{1,-8} \circ \text{id}^3) = (\text{id}^3)^{-1}(\text{domain}(l_{1,1})) \cap (l_{1,-8} \circ \text{id}^3)^{-1}(\text{domain}(\text{recip}))$$

$$= (\text{id}^3)^{-1}(\mathbb{R}) \cap (\text{id}^3)^{-1}((l_{1,-8})^{-1}(\mathbb{R} \setminus \{0\}))$$

$$= (0, \infty) \cap (\text{id}^3)^{-1}(\mathbb{R} \setminus \{8\}) = (0, \infty) \cap (\mathbb{R} \setminus \{2\}) = (0, 2) \cup (2, \infty).$$

The range of R is found like so. We notice

$$R(t) = \frac{t^3 + 1}{t^3 - 8}$$

for all $t \in (0, 2) \cup (2, \infty)$. If $0 < t < 2$, then

$$1 < t^3 + 1 < 9$$

and

$$\frac{1}{t^3 - 8} < -\frac{1}{8}.$$

So,

$$R(t) = \frac{t^3 + 1}{t^3 - 8} = (t^3 + 1) \cdot \frac{1}{t^3 - 8} < -\frac{1}{8}(t^3 + 1) < -\frac{1}{8}.$$

Notice

$$R(t) = 1 + \frac{9}{t^3 - 8}$$

by the division lemma. So, if $t > 2$, then $R(t) > 1$. Combining this, we've proved

$$\text{range}(R) \subseteq (-\infty, -\frac{1}{8}) \cup (1, \infty). \tag{6.8}$$

we conjecture the range of R is

$$(-\infty, -\frac{1}{8}) \cup (1, \infty).$$

We will prove this in the following way.

Finding the inverse of a function is a matter of expressing the input of the function in terms of its output. Again,

$$R(t) = \frac{t^3 + 1}{t^3 - 8}$$

for all $t \in (0, 2) \cup (2, \infty)$. Hence, if $s = R(t)$, then

$$s(t^3 - 8) = t^3 + 1.$$

Hence,

$$st^3 - t^3 = 1 + 8s.$$

Hence,

$$t^3(s - 1) = 1 + 8s.$$

Define

$$Q(s) = t = \left(\frac{8s + 1}{s - 1}\right)^{\frac{1}{3}}$$

for all $s \in (-\infty, -\frac{1}{8}) \cup (1, \infty)$.

We claim Q is the inverse of R. Showing $Q \circ R = \mathrm{id}_{\mathrm{domain}(R)}$ and $R \circ Q = \mathrm{id}_{\mathrm{domain}(Q)}$ will verify that Q is the inverse of R, and is left to the reader.

We instead show $(-\infty, -\frac{1}{8}) \cup (1, \infty)$ is the range of R. If $s < -\frac{1}{8}$, then $8(s - 1) < 8s + 1 < 0$ and $s - 1 < 0$. So,

$$0 < \frac{8s + 1}{s - 1} < 8.$$

Hence, if $s < -\frac{1}{8}$, then

$$0 < Q(s) < 2$$

because $\mathrm{id}^{\frac{1}{3}}$ is increasing on $\mathbb{R}$. So, if

$$s < -\frac{1}{8},$$

then there exists $t \in (0, 2)$ such that $R(t) = s$. Namely,

$$t = Q(s).$$

Since

$$R(Q(s)) = s$$

for all $s \in \mathrm{domain}(Q)$, this proves

$$(-\infty, -\frac{1}{8}) \subseteq \mathrm{range}(R).$$

Similarly, if $s > 1$, then $0 < 8(s - 1) < 8s + 1$. So,

$$8 < \frac{8s + 1}{s - 1}.$$

Hence, if $s > 1$, then

$$2 < Q(s).$$

So, if

$$s > 1,$$

then there exists $t \in (2, \infty)$ such that $R(t) = s$. Namely,

$$t = Q(s).$$

Since

$$R(Q(s))$$

for all $s \in \text{domain}(Q)$, this proves

$$(1, \infty) \subseteq \text{range}(R).$$

Hence,

$$(-\infty, -\frac{1}{8}) \cup (1, \infty) \subseteq \text{range}(R).$$

This, combined with Equation 6.8, proves

$$\text{range}(R) = (-\infty, -\frac{1}{8}) \cup (1, \infty).$$

7 exponential functions

Exercise 7.3. Use the rules of exp and log to demonstrate these identities of their average rate of change.

Solution:

$$\frac{\exp_a(s+t)-\exp_a(s)}{t} = \frac{a^{s+t}-a^s}{t} = \frac{a^s\cdot a^t - a^s}{t} = \frac{a^s(a^t-1)}{t} = a^s\cdot\frac{a^t-1}{t}.$$

$$\begin{aligned}\frac{\log_a(A+B)-\log_a(A)}{B} = \frac{\log_a(A+B)+\log_a(A^{-1})}{B} &= \frac{\log_a((A+B)A^{-1})}{B}\\ &= \frac{1}{B}\cdot\log_a(1+BA^{-1})\\ &= \log_a((1+BA^{-1})^{\frac{1}{B}}).\end{aligned}$$

Exercise 7.4. Suppose a, b are positive, nonone real numbers, and m, n are nonzero real numbers, and k, d are real numbers.

(a) Find the inverse of $l_{n,k} \circ \exp_a \circ\ l_{m,d}$ as a composition involving linear functions.

(b) Use properties of exponents to write $l_{n,k} \circ \exp_a \circ\ l_{m,d}$ as $l_{s,u} \circ \exp_b \circ\ l_{w,v}$ for some real numbers s, u, w and v and where $|s| = 1$.

(c) In the above exercise, find b for which $w = 1$.

Solution:

(a) Recall if f and g are invertible, then their composition $f \circ g$ is, with inverse $g^{-1} \circ f^{-1}$. Since $\exp_a$ is invertible with inverse $\log_a$, and $l_{n,k}$ is invertible with inverse $l_{n^{-1},-n^{-1}k}$, it follows the inverse of $l_{n,k} \circ \exp_a \circ\ l_{m,d}$ is

$$(l_{n,k} \circ \exp_a \circ\ l_{m,d})^{-1} = l_{m,d}^{-1} \circ (l_{n,k} \circ \exp_a)^{-1} = l_{m^{-1},-m^{-1}d} \circ \log_a \circ l_{n^{-1},-n^{-1}k}.$$

(b) Notice $l_{n,k} \circ \exp_a \circ\ l_{m,d}(x) = na^{mx+d} + k$ for all $x \in \mathbb{R}$. So, $l_{n,k} \circ \exp_a \circ\ l_{m,d} = l_{s,u} \circ \exp_b \circ\ l_{w,v}$ if and only if $na^{mx+d} + k = sb^{wx+v} + u$ for all $x \in \mathbb{R}$.

We note $n = \operatorname{sgn}(n)|n| = \operatorname{sgn}(n)a^{\log_a(|n|)}$ since $|n| > 0$ by assumption. Hence,

$$na^{mx+d} = \operatorname{sgn}(n)a^{\log_a(|n|)}a^{mx+d} = \operatorname{sgn}(n)a^{mx+d+\log_a(|n|)}.$$

Also,

$$a^y = b^{\log_b(a)y}$$

for any real y. So,

$$na^{mx+d} = \operatorname{sgn}(n)b^{\log_b(a)(mx+d+\log_a(|n|))}.$$

Hence, if

$$s = \operatorname{sgn}(n),$$

then $|s| = 1$. And if

$$u = k,$$
$$w = m\log_b(a),$$

and

$$v = (\log_a(|n|) + d)\log_b(a),$$

then $na^{mx+d} + k = \operatorname{sgn}(n)\exp_b\left(m\log_b(a)x + (\log_a(|n|) + d)\log_b(a)\right) + k = sb^{wx+v} + u$ for all $x \in \mathbb{R}$.

$$l_{n,k} \circ \exp_a \circ\ l_{m,d} = l_{s,u} \circ \exp_b \circ\ l_{w,v}.$$

(c) We know $w = m\log_b(a)$. So,

$$w = 1$$

if and only if

$$b^{\frac{1}{m}} = a$$

if and only if

$$b = a^m.$$

Exercise 7.5.

(a) Find all real numbers t such that

$$\left(\frac{1}{\log_2 t}\right)^2 + \log_t 8 = 4.$$

(b) Find all real numbers t such that

$$\log_t(2) - \log_4(t) = \log_2(3t).$$

Solution:

(a) We note

$$\log_t(8) = \frac{\log_2(8)}{\log_2(t)} = \frac{3}{\log_2(t)}$$

by the change of base formula and that $2^3 = 8$. Hence,

$$\left(\frac{1}{\log_2 t}\right)^2 + \log_t 8 = 4$$

if and only if

$$\left(\frac{1}{\log_2 t}\right)^2 + \frac{3}{\log_2(t)} - 4 = 0.$$

If $u = \dfrac{1}{\log_2 t}$, then

$$u^2 + 3u - 4 = 0.$$

Hence,

$$\left(\frac{1}{\log_2 t}\right)^2 + \log_t 8 = 4$$

if and only if

$$(u + 4)(u - 1) = 0$$

as long as $u = \dfrac{1}{\log_2 t}$. Hence, by the Zero Product Property, either

$$\frac{1}{\log_2 t} = -4$$

or

$$\frac{1}{\log_2 t} = 1.$$

Hence, either

$$\log_2(t) = -\frac{1}{4}$$

or

$$\log_2(t) = 1.$$

Hence, either

$$t = 2^{-\frac{1}{4}}$$

or

$$t = 2.$$

The verification is left to the reader.

(b) Notice $\log_t(2) = \frac{\log_2(2)}{\log_2(t)} = \frac{1}{\log_2(t)}$, $\log_4(t) = \frac{\log_2(t)}{2}$ and $\log_2(3t) = \log_2(3)+\log_2(t)$ for all positive, nonone t. Hence,

$$\log_t(2) - \log_4(t) = \log_2(3t)$$

if and only if

$$\frac{1}{\log_2(t)} - \frac{\log_2(t)}{2} = \log_2(3) + \log_2(t)$$

if and only if

$$2 - (\log_2(t))^2 = 2\log_2(3)\log_2(t) + 2(\log_2(t))^2$$

if and only if

$$3(\log_2(t))^2 + \log_2(9)\log_2(t) - 2 = 0$$

if and only if

$$\log_2(t) = \frac{\log_2(9) \pm \sqrt{(\log_2(9))^2 + 24}}{6}$$

by the quadratic formula if and only if

$$t = \exp_2\left(\frac{\left(-\log_2(9) \pm \sqrt{(\log_2(9))^2 + 24}\right)}{6}\right).$$

The verification of these solutions is left to the reader.

Exercise 7.6. Suppose b is a real number greater than one. Define

$$f = \exp_b \circ (-\operatorname{recip})|_{(0,\infty)} \cup \operatorname{const}_{(-\infty,0],0}.$$

Recall $-\operatorname{recip}|_{(0,\infty)}(x) = -\frac{1}{x}$ for all $x > 0$, the restriction of the negative reciprocal to $(0,\infty)$. And $\operatorname{const}_{(-\infty,0],0}(x) = 0$ if $x \leq 0$, the constant zero function on $(-\infty, 0]$.

(a) What is the domain of f?

(b) What is the range of f?

(c) Is f invertible? If not, explain why.

(d) What is the largest interval, I, on which f is one-to-one? Find the inverse of $f|_I$.

(e) What is the global data of f? That is, does f have any limits at infinity?

(f) Sketch the graph of f.

Solution:

(a) The domain of f is $\mathbb{R}$, since it is the union of a function with domain $(0,\infty)$ with a function with domain $[0,\infty)$.

(b) We claim the range of f is $[0,1)$. To prove this, we note, $\exp_b(\mathbb{R}) = (0,\infty)$. Since $b > 1$, $\exp_b$ is increasing. And $\exp_b(0) = 1$. Hence, $\exp_e((-\infty,0)) = (0,1)$. If $x > 0$, then $-\frac{1}{x} < 0$ and $f(x) = e^{-\frac{1}{x}}$. Hence,

$$f(x) \in (0,1).$$

Hence,

$$f((0,\infty)) \subseteq \exp_b \circ (-\operatorname{recip})|_{(0,\infty)}((0,\infty)) \subseteq (0,1).$$

If $y \in (0,1)$, then there is some $x > 0$ such that $f(x) = y$. Namely, $x = -\frac{1}{\log_b(y)}$. Hence, $(0,1) \subseteq \operatorname{range}(f)$. And

$$f((-\infty,0]) = \{0\},$$

since f is constant there. Hence,

$$\operatorname{range}(f) = [0,1).$$

(c) f is not invertible because f is not one-to-one. For example, $f(-1) = 0 = f(0)$.

(d) Let $I = [0,\infty)$. $f|_I$ is one-to-one, as noted by part (b). Define

$$g = -\operatorname{recip} \circ \log_b|_{(0,1)} \cup \{(0,0)\}.$$

We claim g is the inverse of $f|_I$.

if $0 < y < 1$, then $g(y) = -\frac{1}{\log_b(y)} > 0$ and

$$f|_I(g(y)) = \exp_b\left(-\frac{1}{-\frac{1}{\log_b(y)}}\right) = \exp_b(\log_b(y)) = y.$$

And $f(g(0)) = 0$.

Similarly, if $x > 0$, then $0 < f|_I(x) < 1$ and

$$g(f|_I(x)) = -\frac{1}{\log_b(\exp_b(-\frac{1}{x}))} = -\frac{1}{-\frac{1}{x}} = x.$$

And $g(f(0)) = 0$. Hence,

$$f|_I \circ g = \mathrm{id}_{\mathrm{domain}(g)}$$

and

$$g \circ f|_I = \mathrm{id}_{\mathrm{domain}(f|_I)}.$$

Any other interval larger than I includes negative numbers. And we know f is not one-to-one on any interval containing 0 and a negative number.

(e) We prove

$$\lim_{\infty} f = 1$$

and

$$\lim_{-\infty} f = 0.$$

$$\lim_{\infty} f = \lim_{\infty} \exp_b \circ(-\,\mathrm{recip}) = \exp_b \circ(\lim_{\infty}(-\,\mathrm{recip})) = \exp_b(0) = 1.$$

The second step follows from the continuity of $\exp_b$. And

$$\lim_{-\infty} f = \lim_{-\infty} \mathrm{const}_0 = 0.$$

(f)