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Some notes on some partial differential equations

1 First order PDE

Theorem 1.1. Existence and uniqueness of quasi-linear first order PDE. [Folland, '95]

Suppose $n \in \mathbb{N}$, $n > 1$, S is a hypersurface of $\mathbb{R}^n$, $\mathcal{O} \subseteq \mathbb{R}^n$ is open, contains S as a subset, $A \in C^1(\mathcal{O} \times \mathbb{R}, \mathbb{R}^{n+1})$ and $\phi \in C^1(\mathcal{O})$, such that if $X_p \in T_p\mathbb{R}^n$ is defined as

$$X_p = \sum_{i=1}^{n} A \circ ((\mathrm{id}_{\mathcal{O}}, \phi))(p)(i) \left.\frac{\partial}{\partial r^i}\right|_p$$

for all $p \in \mathcal{O}$, then $X_p \notin D(\iota_{S,\mathbb{R}^n})(p)(T_pS)$ for all $p \in S$. Then there is some open set $\Omega \subseteq \mathbb{R}^n$ such that $S \subset \Omega \subseteq \mathcal{O}$ and a unique solution, $u \in C^1(\Omega)$, of

$$\begin{cases} D(v)(B) = 0 & \text{on} \quad \Omega \times \mathbb{R} \\ u = \phi & \text{on} \quad S. \end{cases} \tag{1.2}$$

Here,

$$v = u \circ P^{\{1.n\}} - r^{n+1},$$
$$B = A \circ (P^{\{1.n\}}, u \circ P^{\{1.n\}})$$

on $\Omega \times \mathbb{R}$, $D(v)(B)$ is a function as described in Definition C.3 and $(\mathrm{id}_{\mathcal{O}}, \phi)$ and so on is a rectangular product of functions as described in Definition A.1.

Proof. Fix $p \in S$ and an adapted chart $(U \cap S, \varphi_S)$ of (U, φ), a chart of $\mathbb{R}^n$, relative to S with $p \in U$. Assume further U is chosen such that $U \subseteq \mathcal{O}$ and φ, ϕ are bounded on U. Then there is some $I \subset \{1.n\}$ of size $n-1$ such that

$$\varphi(U \cap S) = \varphi(U) \cap \left(P^{\{1.n\}\setminus I}\right)^{-1}(\{0\}).$$

Now if $j \in \{1.n\} \setminus I$, define a function $\psi(x)(i) = x(i)$ if $i \neq j, n$, $\psi(x)(j) = x(n)$ and $\psi(x)(n) = x(j)$. Then ψ is a diffeomorphism on $\mathbb{R}^n$ and

$$\psi\left(\left(P^{\{1.n\}\setminus I}\right)^{-1}(\{0\})\right) = \mathbb{R}^{n-1}.$$

Then $(U \cap S, (\psi \circ \varphi)_S)$ is an adapted chart relative to S and $\psi \circ \varphi(U \cap S) = \psi \circ \varphi(U) \cap \mathbb{R}^{n-1}$.

Let $\eta = \psi \circ \varphi$ and $V = \eta(U) \cap \mathbb{R}^{n-1}$.

There exists J, open in $\mathbb{R}$, containing 0 and a unique $y \in C^1(V \times J, \mathbb{R}^{n+1})$ such that

$$\begin{cases} \frac{\partial}{\partial r^n} y = A \circ y & \text{on} \quad V \times J \\ y = (\eta^{-1}, \phi \circ \eta^{-1}) & \text{on} \quad V \times \{0\}. \end{cases} \tag{1.3}$$

by Theorem F.1. This initial condition, the condition on X_p, the fact that invertibility is an open condition and the inverse function theorem implies the map

$$x = P^{\{1.n\}} \circ y$$

is a diffeomorphism on an open subset $W \subseteq V \times J$ containing $V \times \{0\}$. To note this, if $i \in \{1.n-1\}$, $v \in V$, then

$$Dx((v,0))(\left.\frac{\partial}{\partial r^i}\right|_p) = P^{\{1.n\}} \circ D((\eta^{-1}, (\phi \circ \eta^{-1})) \circ P^{\{1.n-1\}})((v,0))(\left.\frac{\partial}{\partial r^i}\right|_p)$$
$$= D\eta^{-1}(v)(\left.\frac{\partial}{\partial r^i}\right|_p)$$

by Theorem ??. Hence, by Theorem E.10, the range of $Dx((v,0))$ restricted to $T_p\mathbb{R}^{n-1}$ is $D(\iota_{S,\mathbb{R}^n})(p)(T_pS)$, which is a $n-1$ dimensional vector space if $\eta(p)=v$.

$$Dx((v,0))(\frac{\partial}{\partial r^n}\bigg|_p) = P^{\{1.n\}}\circ Dy((v,0))(\frac{\partial}{\partial r^n}\bigg|_p) = P^{\{1.n\}}\circ\frac{\partial}{\partial r^n}y((v,0))$$
$$= P^{\{1.n\}}\circ A\circ y((v,0))$$
$$= P^{\{1.n\}}\circ A\circ(\eta^{-1}(v),\phi(\eta^{-1}(v))).$$

Hence, by our condition on X_p, the range of $Dx((v,0))$ is $T_p\mathbb{R}^n$. Hence, since its domain is as well, it's invertible. The argument now follows from the inverse function theorem.

Define

$$u_p = r^{n+1}\circ y\circ x^{-1} \quad \text{on} \quad x(W).$$

Then

$$B = A\circ y\circ x^{-1}\circ P^{\{1.n\}}$$

and

$$v = r^{n+1}\circ y\circ x^{-1}\circ P^{\{1.n\}} - r^{n+1}$$

on $x(W)\times\mathbb{R}$. Notice

$$A\circ y(w) = \frac{\partial}{\partial r^n}y(w) = Dy(w)(\mathbf{e}_n)$$

for all $w\in W$.

Hence, for any $(q,t)\in x(W)\times\mathbb{R}$, if $w=x^{-1}(q)$, then, by the Chain rule

$$\begin{aligned}Dv(B)(q,t) &= \left(r^{n+1}\circ D(y\circ x^{-1})(q)\circ P^{\{1.n\}} - r^{n+1}\right)(A\circ y(w))\\ &= r^{n+1}\circ Dy(w)Dx^{-1}(q)\circ P^{\{1.n\}}(Dy(w)(\mathbf{e}_n)) - r^{n+1}\circ Dy(w)(\mathbf{e}_n)\\ &= r^{n+1}\circ Dy(w)D(x^{-1}\circ x)(w)(\mathbf{e}_n)) - r^{n+1}\circ Dy(w)(\mathbf{e}_n)\\ &= r^{n+1}\circ Dy(w)(\mathbf{e}_n) - r^{n+1}\circ Dy(w)(\mathbf{e}_n)\\ &= 0.\end{aligned}$$

And if $q\in S\cap x(W)$, then $\eta(q)\in V\times\{0\}\subseteq W$. Hence, $\eta(q)=x^{-1}(q)$ and $y(x^{-1}(q))=(q,\phi(q))$. Hence,

$$u_p(q) = r^{n+1}(y(x^{-1}(q))) = \phi(q).$$

To prove uniqueness, suppose u solves 1.2 on $x(W)$. Then there exists $\tilde{W}\subseteq V\times J$ and a unique $h\in C^1(\tilde{W},\mathbb{R}^n)$ such that

$$\begin{cases}\frac{\partial}{\partial r^n}h = P^{\{1.n\}}\circ A\circ(h,u\circ h) & \text{on} \quad \tilde{W}\\ h=\eta^{-1} & \text{on} \quad \tilde{W}\cap(r^n)^{-1}(\{0\}).\end{cases}$$

Then define $\tilde{y}=(h,u\circ h)$. Then $\tilde{y}\in C^1(\tilde{W},\mathbb{R}^{n+1})$,

$$\begin{aligned}\frac{\partial}{\partial r^n}\tilde{y} &= (P^{\{1.n\}}\circ A\circ\tilde{y}, Du\circ P^{\{1.n\}}\circ A\circ\tilde{y})\\ &= (P^{\{1.n\}}\circ A\circ\tilde{y}, r^{n+1}\circ A\circ\tilde{y})\\ &= A\circ\tilde{y}\end{aligned}$$

and if $w \in \tilde{W} \cap (r^n)^{-1}(\{0\})$, then $\eta^{-1}(w) \in S$, so that

$$\tilde{y}(w) = (\eta^{-1}(w), u \circ \eta^{-1}(w)) = (\eta^{-1}(w), \phi \circ \eta^{-1}(w)).$$

Hence, by the uniqueness of y, $\tilde{y} = y$ on $W \cap \tilde{W}$. Hence, $h = x$ and

$$u = r^{n+1} \circ \tilde{y} \circ x^{-1} = u_p$$

on $x(W \cap \tilde{W})$.

Now define $\Omega_p = x(W)$. By the above uniqueness argument, if $q \in S$ as well, $u_p = u_q$ on $\Omega_p \cap \Omega_q$.

Now, define

$$\Omega = \bigcup \{\Omega_p \in \mathcal{P}(\mathbb{R}^n) \mid p \in S\}.$$

Then Ω is open in $\mathbb{R}^n$ and contains S. Define, u on Ω such that, for all $p \in S$,

$$u|_{\Omega_p} = u_p.$$

Then u solves 1.2 and is the unique such function on Ω which does so.

□

Example 1.4. [Folland, '95] There exists an open $\Omega \subseteq \mathbb{R}^n$ and a unique solution, u, of

$$u\frac{\partial}{\partial r^1}(u) + \frac{\partial}{\partial r^2}(u) = 1 \qquad \text{on} \qquad \Omega.$$

with

$$u = \frac{1}{2}x \qquad \text{on} \qquad \{x = y\} \cap \{|x + y| < 2\}.$$

Proof. If $\gamma = x - y$, then $\nabla\gamma = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and if $A = \begin{pmatrix} u \\ 1 \end{pmatrix}$, then on

$$\{x = y\} \cap \{|x + y| < 2\} = \gamma^{-1}(\{0\}) \cap \{|x + y| < 2\},$$

$$\nabla\gamma \cdot A = \frac{1}{2}x - 1 \neq 0$$

since $-1 < x < 1$.

A parameterization of $\gamma^{-1}(\{0\}) \cap \{|x + y| < 2\}$ is given by $s \mapsto s(\mathbf{e}_1 + \mathbf{e}_2)$ when $-1 < s < 1$. Let's find the characteristic curves of the system

$$\begin{cases} \mathbf{Z}_t(s,t) = \begin{pmatrix} w(s,t) \\ 1 \end{pmatrix} & \text{if} \quad (s,t) \in (-1,1) \times \mathbb{R} \\ w_t(s,t) = 1 & \text{if} \quad (s,t) \in (-1,1) \times \mathbb{R} \\ \mathbf{Z}(s,0) = s(\mathbf{e}_1 + \mathbf{e}_2) & \text{if} \quad -1 < s < 1 \\ w(s,0) = \frac{1}{2}s & \text{if} \quad -1 < s < 1 \end{cases}$$

(compare with (**??**)).

We notice $w(s,t) = t + \frac{1}{2}s$. So,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{Z}(s,t) = \begin{pmatrix} \frac{1}{2}t^2 + \frac{1}{2}st + s \\ t + s \end{pmatrix}$$

Inverting **Z** (solving for s and t), we obtain

$$s = \frac{y^2 - 2x}{y - 2}$$

and

$$t = \frac{2(x - y)}{y - 2}$$

Keeping in mind $w = u \circ \mathbf{Z}$, we have

$$u = \frac{y^2 + 2x - 4y}{2(y - 2)}.$$

Define $\Omega = \{y < 2\}$. □

2 Auxiliary conditions and second order equations

Auxiliary conditions and second order equations

Usually, if there is a solution of a PDE, then there are many. In practice, a PDE is considered alongside an auxiliary condition, as in Theorem **??**. An auxiliary condition is usually referred to as an **initial** or **boundary** condition, depending on the physics.

For instance, (1.2), which is

$$\begin{cases} a\frac{\partial}{\partial r^1}(u) + b\frac{\partial}{\partial r^2}(u) + cu = f & \text{on} \quad \Omega \\ u = g \circ x & \text{on} \quad y = 0. \end{cases}$$

might be considered as an initial value problem if the y variable is treated as time, with $g \circ x$ the initial condition. Many types of initial and boundary conditions are found in [Strauss, '92 pg 20-24], [Haberman, '04 pg 12-18], and [Haberman, '04 pg 139-141].

From [Strauss, '92 pg 25-26]. Auxiliary conditions are usually required for a problem to be **well-posed**.

Definition 2.1. A PDE together with some auxiliary conditions is said to be **well-posed** if

1. (existence) There is a solution of the PDE which satisfies the auxiliary conditions.
2. (uniqueness) There is at most one solution of the problem.
3. (stability) The solution depends continuously on the data of the problem (e.g., the coefficients of the PDE, the auxiliary conditions, etc.)

We've noted (in Theorem **??**) quasilinear first order PDE in two variables have unique solutions given a reasonable auxiliary condition (depending on the PDE). In fact, in a certain sense, these problems are well-posed. Compare exercise 4 in homework 2.

The equation

$$u_{xx} = 0$$

on $\mathbb{R}^n$ is an example of a **second order** PDE, since it nontrivially involves a second partial derivative of the unknown function. A solution is given by $x g \circ y + h \circ y$, where g and h are given functions of a single variable. What auxiliary conditions grant existence and uniqueness? Compare exercise 5 in homework 2.

From [Strauss, '92 pg 25-26]. Stability is probably the least familiar of these conditions, and it requires some formulation, depending on the notion of nearness of functions, which we will hopefully get to for various PDE. For now, consider the following. An important example of a second order PDE is **Laplace's equation in the plane**, given by

$$u_{xx} + u_{yy} = 0.$$

Consider the open set in the plane described by $y > 0$, called the upper-half plane, denoted by H. That is, $H = \{\mathbf{z} \in \mathbb{R}^n \mid y(\mathbf{z}) > 0\}$.

We say the **boundary** of the upper-half plane is given by $y = 0$, since for each $\mathbf{z}$ on $y = 0$, for any disk, D, containing $\mathbf{z}$, there are points $w \in D \cap H$ and points $w \in D \cap (\mathbb{R}^n \setminus H)$. The x-axis, $y = 0$, is denoted by bdy H in this case. On H, for any positive integer n, a solution of Laplace's equation is given by

$$u_n = \frac{1}{n} e^{-\sqrt{n}} \sin \circ nx \sinh \circ ny.$$

(Here u_n does not refer to partial differentiation of some function u with respect to n.)

Then

$$u_n = 0 \quad \text{on} \quad \text{bdy } H$$

and

$$\frac{\partial}{\partial r^2} u_n = e^{-\sqrt{n}} \sin \circ nx \quad \text{on} \quad \text{bdy } H.$$

We see, if $\mathbf{z} \in$ bdy H, then

$$u_n(\mathbf{z}) = 0 \to 0$$

and

$$\frac{\partial}{\partial r^2} u_n(\mathbf{z}) = e^{-\sqrt{n}} \sin(nx(\mathbf{z})) \to 0$$

as $n \to \infty$. But for any $\mathbf{z} \in H$, the limit of $u_n(\mathbf{z})$ is ∞ as n approaches ∞. So, stability is not guaranteed even if existence and uniqueness are. (In this case, the size of the auxiliary condition does not control the size of the solutions.)

Recall Laplace's equation in the plane:

$$u_{xx} + u_{yy} = 0.$$

We introduce related second order equations, namely the wave and heat equations.

The **1-dimensional wave equation** is given by

$$u_{tt} - c^2 u_{xx} = 0,$$

where c is a positive constant, known as the wave propagation speed. Refer to [Haberman, '04 pg 135-138], [Strauss, '92 pg 20-24], or [Weinberger, '65 pg 1-8] for derivations. The 1-d wave equation is an equation of two variables. The unknown is a function of x and t (instead of y), and u_{tt} denotes the second partial derivative of u with respect to t. Physically, u represents the displacement of a vibrating string from the horizontal.

The **1-dimensional heat equation** is given by

$$u_t - k u_{xx} = 0,$$

where k is a positive constant, known as a proportionality constant relating the specific heat, density, and heat conductivity of the object. Refer to [Haberman, '04 pg 2-10], [Hancock, '06 pg 1-3], or [Strauss, '92 pg

20-24] for derivations. Physically, u represents the distribution of heat in a thin rod. This simplified model of heat distribution also models the concentration of a substance in a pipe, so is also referred to as the **diffusion equation**.

Laplace's equation, the wave equation and the heat equation are all **homogeneous linear** second order equations. Homogeneous linear PDE enjoy the **superposition principle**. That is, linear combinations of solutions (over $\mathbb{R}$) are also solutions.

From [Strauss, '92 pg 82-86]. We consider the wave and heat equation on the domain

$$(0, l) \times (0, T) \subseteq \mathbb{R}^n,$$

with some auxiliary conditions, where l and T are both fixed positive constants.

In particular, suppose g and h are functions on $[0, l]$. Consider

$$u_{tt} = c^2 u_{xx} \qquad \text{on} \qquad (0, l) \times (0, T),$$

with the **Dirichlet** boundary conditions (u is prescribed at the boundary)

$$u(0, t) = 0 = u(l, t) \qquad \text{on} \qquad [0, T],$$

and the initial conditions

$$u(x, 0) = g(x) \qquad \text{on} \qquad [0, l],$$
$$u_t(x, 0) = h(x) \qquad \text{on} \qquad [0, l].$$

That is, consider the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0, l) \times (0, T) \\ u(0, t) = 0 = u(l, t) & \text{on} \quad [0, T] \\ u(x, 0) = g(x) & \text{on} \quad [0, l] \\ u_t(x, 0) = h(x) & \text{on} \quad [0, l]. \end{cases} \tag{2.2}$$

This problem is well-posed for most functions g and h. For us, this will not be clear until we discuss Fourier series next week.

We make the ansatz (educated guess) to separate the variables of u. We consider $u(x, t) = \phi(x)\psi(t)$, where ϕ and ψ are functions of a single variable. That is, we consider a solution of the wave equation in two variables which is the product of two functions in one variable. If we find many such solutions, we can employ the superposition principle and hopefully construct a function which solves (2.2).

If

$$u_{tt} = c^2 u_{xx}$$

on $(0, l) \times (0, T)$ and

$$u(x, t) = \phi(x)\psi(t)$$

on $[0, l] \times [0, T]$, then

$$\phi(x)\psi''(t) = c^2 \phi''(x)\psi(t) \qquad \text{on } (0, l) \times (0, T), \tag{2.3}$$

where $'$ denotes differentiation with respect to one variable. If $u \neq 0$ as functions on $(0, l) \times (0, T)$, then there is some $(x_0, t_0) \in (0, l) \times (0, T)$ such that $\phi(x_0) \neq 0$ and $\psi(t_0) \neq 0$.

Let $\lambda_1 = -\frac{\phi''(x_0)}{\phi(x_0)}$ and $\lambda_2 = -\frac{\psi''(t_0)}{c^2\psi(t_0)}$. Then, from (2.3),

$$\begin{cases} \psi''(t) = -c^2\lambda_1\psi(t) & \text{on } (0,T) \\ \phi''(x) = -\lambda_2\phi(x) & \text{on } (0,l) \end{cases}$$

Hence,

$$\begin{cases} \phi(x)\psi''(t) = -c^2\lambda_1 u(x,t) & \text{on } (0,l)\times(0,T) \\ c^2\phi''(x)\psi(t) = -c^2\lambda_2 u(x,t) & \text{on } (0,l)\times(0,T) \end{cases}$$

The left hand side of these equations are equal on this rectangle from (2.3). Hence,

$$c^2(\lambda_1 - \lambda_2)u = 0$$

on $(0,l)\times(0,T)$. Since $c > 0$ and u is not identically zero,

$$\lambda_1 = \lambda_2.$$

Denote this quantity by λ. To conclude, if $u(x,t) = \phi(x)\psi(t)$ is a solution of the 1-d wave equation, then necessarily

$$\begin{cases} \psi''(t) = -c^2\lambda\psi(t) & \text{on } (0,T) \\ \phi''(x) = -\lambda\phi(x) & \text{on } (0,l) \end{cases}$$

for some constant λ.

Proposition 2.4. If $u(x,t) = \phi(x)\psi(t)$ is a nonzero solution of (2.2), then $\lambda > 0$.

Proof. The initial conditions $u(0,t) = u(l,t) = 0$ on $[0,T]$ imply $\phi(0) = \phi(l) = 0$. Otherwise, u is identically zero.

If $\lambda = 0$, then $\phi'' = 0$, from which $\phi = ax + b$ for some $a, b \in \mathbb{R}$ on $(0,l)$ follows. Since $\phi(0) = \phi(l) = 0$, it follows $\phi = 0$ on $[0,l]$. Then u is identically zero, which is a contradiction.

If $\lambda < 0$, then $\phi'' = -\lambda\phi$ implies $\phi(x) = a\cosh(\gamma x) + b\sinh(\gamma x)$ for some $a, b \in \mathbb{R}$. Here, $\gamma = \sqrt{-\lambda}$. Since $\sinh(0) = 0$ and $\cosh(0) = 1$, $\phi(0) = 0$ implies $a = 0$. Since $\sinh(\gamma l) > 0$, $\phi(l) = 0$ implies $b = 0$. Then, again, $\phi = 0$ on $[0,l]$, which implies $u = 0$ on $[0,l]\times[0,T]$, a contradiction. By trichotomy, $\lambda > 0$. ■

Recall the 1-d wave equation $u_{tt} = c^2u_{xx}$ on $(0,l)\times(0,T)$, together with the spatial Dirichlet boundary condition $u(0,t) = 0 = u(l,t)$ on $[0,T]$, reduces to a pair of ODE with auxiliary conditions:

$$\begin{cases} \psi''(t) = -c^2\lambda\psi(t) & \text{on } (0,T) \\ \phi''(x) = -\lambda\phi(x) & \text{on } (0,l) \\ \phi(0) = \phi(l) = 0 \end{cases}$$

for some constant $\lambda > 0$, if $u(x,t) = \phi(x)\psi(t)$. Since $\lambda > 0$, $\lambda = \beta^2$ for some $\beta \in \mathbb{R}$. Then there are some $p, q \in \mathbb{R}$ such that

$$\phi(x) = p\cos(\beta x) + q\sin(\beta x)$$

and some $a, b \in \mathbb{R}$ such that

$$\psi(t) = a\cos(c\beta t) + b\sin(c\beta t).$$

Now we again employ the boundary condition $\phi(0) = \phi(l) = 0$ to conclude $p = 0$. If u is not identically zero, $q \neq 0$, so that

$$\beta = \frac{n\pi}{l}$$

for some integer n, since $q\sin(l) = 0$.

Hence,

$$u(x,t) = \left(a\cos\left(c\frac{n\pi}{l}t\right) + b\sin\left(c\frac{n\pi}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right).$$

Algebraically, we can say the set of eigenvalues of the operator $-\frac{d^2}{dr^2}$, restricted to say functions whose second derivative is continuous on $(0, l)$ and which satisfy the homogeneous Dirichlet boundary condition, is the countable set

$$\left\{\left(\frac{n\pi}{l}\right)^2 \in \mathbb{R} \mid n \in \mathbb{Z}\right\}.$$

That is, if

$$\begin{cases} -\phi''(x) = \lambda\phi(x) & \text{on } (0,l) \\ \phi(0) = \phi(l) = 0 \end{cases}$$

for some ϕ, then

$$\lambda \in \left\{\left(\frac{n\pi}{l}\right)^2 \in \mathbb{R} \mid n \in \mathbb{Z}\right\}.$$

Recall we wish to solve the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0,l)\times(0,T) \\ u(0,t) = 0 = u(l,t) & \text{on} \quad [0,T] \\ u(x,0) = g(x) & \text{on} \quad [0,l] \\ u_t(x,0) = h(x) & \text{on} \quad [0,l]. \end{cases}$$

We've shown if $n > 0$ is an integer, $a_n, b_n \in \mathbb{R}$, then

$$u_n(x,t) = \left(a_n\cos\left(c\frac{n\pi}{l}t\right) + b_n\sin\left(c\frac{n\pi}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)$$

satisfies the first two lines of the above problem. (From the parity of cos and sin, $n < 0$ is redundant, since a_n, b_n are arbitrary.)

Now we employ the superposition principle (finite linear combinations of solutions are solutions) of the wave equation to conclude if n is a positive integer and $a_1.a_n, b_1.b_n \in \mathbb{R}$, then

$$u(x,t) = \sum_{j=1}^{n}\left(a_j\cos\left(c\frac{j\pi}{l}t\right) + b_j\sin\left(c\frac{j\pi}{l}t\right)\right)\sin\left(\frac{j\pi}{l}x\right)$$

solves

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0,l)\times(-\infty,\infty) \\ u(0,t) = 0 = u(l,t) & \text{on} \quad (-\infty,\infty) \end{cases}$$

So that, if

$$g(x) = \sum_{j=1}^{n} a_j \sin\left(\frac{j\pi}{l}x\right) \tag{2.5}$$

and

$$h(x) = c\sum_{j=1}^{n} \frac{j\pi}{l} b_j \sin\left(\frac{j\pi}{l}x\right) \tag{2.6}$$

then

$$u(x,t) = \sum_{j=1}^{n} \left(a_j \cos\left(c\frac{j\pi}{l}t\right) + b_j \sin\left(c\frac{j\pi}{l}t\right)\right) \sin\left(\frac{j\pi}{l}x\right)$$

is a solution of

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0,l)\times(-\infty,\infty) \\ u(0,t) = 0 = u(l,t) & \text{on} \quad (-\infty,\infty) \\ u(x,0) = g(x) & \text{on} \quad [0,l] \\ u_t(x,0) = h(x) & \text{on} \quad [0,l]. \end{cases}$$

Most functions of a single variable do not resemble functions given by (2.5) and (2.6). However, there is a very large class of functions which can be written as *series*

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{l}x\right)$$

and

$$h(x) = c\sum_{n=1}^{\infty} \frac{n\pi}{l} b_n \sin\left(\frac{n\pi}{l}x\right)$$

So, if we consider the series

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(c\frac{n\pi}{l}t\right) + b_n \sin\left(c\frac{n\pi}{l}t\right)\right) \sin\left(\frac{n\pi}{l}x\right),$$

and assume u converges and can be differentiated term by term, then u is a solution of the 1-d wave equation on a rectangle with homogeneous Dirichlet boundary conditions and some very general initial conditions.

Example 2.7. Consider the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0,l)\times(-\infty,\infty) \\ u(0,t) = 0 = u(l,t) & \text{on} \quad (-\infty,\infty) \\ u(x,0) = \sin(\frac{\pi}{l}x) - 2\sin(\frac{2\pi}{l}x) & \text{on} \quad [0,l] \\ u_t(x,0) = \sin(\frac{\pi}{l}x) & \text{on} \quad [0,l]. \end{cases} \tag{2.8}$$

We look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(c\frac{n\pi}{l}t\right) + b_n \sin\left(c\frac{n\pi}{l}t\right)\right) \sin\left(\frac{n\pi}{l}x\right)$$

for some sequences $\{a_n\}, \{b_n\}$.

When $t = 0$,

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{l}x\right) = u(x,0) = \sin\left(\frac{\pi}{l}x\right) - 2\sin\left(\frac{2\pi}{l}x\right)$$

and

$$c\sum_{n=1}^{\infty} \frac{n\pi}{l} b_n \sin\left(\frac{n\pi}{l}x\right) = u_t(x,0) = \sin\left(\frac{\pi}{l}x\right)$$

so that if $a_1 = 1$, $a_2 = -2$ and $a_n = 0$ if $n > 2$, and if $b_1 = \frac{l}{c\pi}$ $b_n = 0$ if $n > 1$, then

$$u(x,t) = \left(\cos\left(c\frac{\pi}{l}t\right) + \frac{l}{c\pi}\sin\left(c\frac{\pi}{l}t\right)\right)\sin\left(\frac{\pi}{l}x\right) \\ -2\cos\left(c\frac{2\pi}{l}t\right)\sin\left(\frac{2\pi}{l}x\right)$$

is a solution of (2.8).

2.1 Heat equation on a rectangle with Dirichlet boundary conditions

The heat equation on $(0,l)\times(0,T)$ with homogeneous Dirichlet boundary conditions can be solved in the same way. If u is a solution of

$$\begin{cases} u_t = ku_{xx} & \text{on} \quad (0,l)\times(0,T) \\ u(0,t) = 0 = u(l,t) & \text{on} \quad [0,T], \end{cases}$$

$u(x,t) = \phi(x)\psi(t)$, and if u is not identically zero, then

$$\begin{cases} \psi'(t) = -k\lambda\psi(t) & \text{on } (0,T) \\ \phi''(x) = -\lambda\phi(x) & \text{on } (0,l) \\ \phi(0) = \phi(l) = 0 \end{cases}$$

for some positive λ.

Notice the problem

$$\begin{cases} \phi''(x) = -\lambda\phi(x) & \text{on } (0,l) \\ \phi(0) = \phi(l) = 0 \end{cases}$$

is exactly the same as the spatial problem for the separated wave equation, which is why we conclude $\lambda = \left(\frac{n\pi}{l}\right)^2$ for some positive integer n. Then a solution of $\psi'(t) = -k\lambda\psi(t)$ is

$$\psi(t) = ae^{-k(n\pi/l)^2 t}$$

if $a \in \mathbb{R}$.

So, if $g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{l}x\right)$, then

$$u = \sum_{n=1}^{\infty} a_n e^{-k(n\pi/l)^2 t}\sin\left(\frac{n\pi}{l}x\right)$$

is a solution of the 1-d heat equation with spatial Dirichlet boundary condition and initial condition

$$\begin{cases} u_t = ku_{xx} & \text{on} \quad (0,l)\times(0,\infty) \\ u(0,t) = 0 = u(l,t) & \text{on} \quad [0,\infty] \\ u(x,0) = g(x) & \text{on} \quad [0,l] \end{cases} \tag{2.9}$$

provided of course u has nice convergence and differentiation properties.

Example 2.10. Consider the problem

$$\begin{cases} u_t = ku_{xx} & \text{on} \quad (0,1)\times(0,T) \\ u(0,t) = 0 = u(1,t) & \text{on} \quad [0,T] \\ u(x,0) = \cos(\pi x) & \text{on} \quad [0,1] \end{cases} \tag{2.11}$$

Again we look for solutions of the form

$$u = \sum_{n=1}^{\infty} a_n e^{-k(n\pi)^2 t} \sin(n\pi x)$$

for some sequence $\{a_n\}$.

Then

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = u(x,0) = \cos(\pi x).$$

How do we solve for a_n? We make the following observations.

$$\int_0^1 \sin(n\pi x)\cos(\pi x)dx = \begin{cases} \frac{2n}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Therefore,

$$a_n = \begin{cases} \frac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if n is odd} \end{cases}$$

if we can integrate $\sum_{n=1}^{\infty} a_n \sin(n\pi x)$ term by term.

So,

$$u = \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} e^{-k(2n\pi)^2 t} \sin(2n\pi x)$$

is a solution of (2.11).

2.2 Wave and heat equations on a rectangle with Neumann boundary conditions

Wave and heat equations on a rectangle with Neumann boundary conditions Let us again consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0,l)\times(0,T) \\ \frac{\partial}{\partial r^1}(u)(0,t) = 0 = \frac{\partial}{\partial r^1}(u)(l,t) & \text{on} \quad [0,T] \\ u(x,0) = g(x) & \text{on} \quad [0,l] \\ u_t(x,0) = h(x) & \text{on} \quad [0,l]. \end{cases} \tag{2.12}$$

but this time with Neumann spatial boundary conditions (prescribed spatial velocity at the endpoints). This means the endpoints of the string is allowed to move in the vertical direction over time, but its displacement is always horizontal there. Refer to [Haberman, '04 pg 139-141] for some more physically motivated boundary conditions.

If $u(x,t) = \phi(x)\psi(t)$ is nonzero, then, ϕ is nonzero and, just as in the Dirichlet case, it follows

$$\begin{cases} \psi''(t) = -c^2\lambda\psi(t) & \text{on } (0,T) \\ \phi''(x) = -\lambda\phi(x) & \text{on } (0,l) \\ \phi'(0) = \phi'(l) = 0 \end{cases}$$

for some λ, possibly complex.

Proposition 2.13.

$$\lambda \in \left\{ \left(\frac{n\pi}{l}\right)^2 \mid n \text{ is a nonnegative integer} \right\}.$$

Proof:

If λ is complex and nonzero, let $\gamma \neq 0$ be one of the square roots of $-\lambda$. then

$$\phi(x) = Ce^{\gamma x} + De^{-\gamma x}$$

for some constants C, D, not both zero. Then $\phi'(x) = \gamma Ce^{\gamma x} - \gamma De^{-\gamma x}$. Since $\phi'(0) = 0$, $0 = \gamma(C - D)$. Therefore, either $\gamma = 0$ or $C = D$.

Since $\lambda \neq 0$, then since $C = D$ and $C \neq 0$, $\phi'(l) = 0$ implies $1 = e^{2l\gamma}$. Hence, $2l\gamma = 2n\pi i$ for some nonzero $n \in \mathbb{Z}$, as seen by Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, where i is the imaginary unit such that $i^2 = -1$. Hence,

$$\lambda = -\gamma^2 = \left(\frac{n\pi}{l}\right)^2.$$

If $\lambda = 0$, then $\phi(x) = Cx + D$ for some C, D not both zero. Then the boundary conditions imply $C = 0$, but D can be arbitrarily nonzero. Hence, $\lambda = 0$ is also an eigenvalue of this problem.

■

What are the eigenvectors of

$$\begin{cases} \phi''(x) = -\lambda\phi(x) & \text{on } (0,l) \\ \phi'(0) = \phi'(l) = 0? \end{cases}$$

Suppose $\lambda \neq 0$, then $\lambda > 0$, so that

$$\phi(x) = C\cos(\beta x) + D\sin(\beta x)$$

for some real C, D, with $\beta = \sqrt{\lambda} > 0$.

$\phi'(x) = -C\beta\sin(\beta x) + D\beta\cos(\beta x)$. Then $0 = \phi'(0) = D\beta$. So, $D = 0$. Hence, $\phi(x) = C\cos(\beta x)$ for some nonzero C.

If $\lambda = 0$, we've established $\phi(x) = D$ for some nonzero D.

Hence, an eigenvector associated with $\left(\frac{n\pi}{l}\right)^2$ is

$$\cos\left(\frac{n\pi}{l}x\right)$$

if $n \neq 0$ and

$$1$$

if $n = 0$.

Since the problem

$$\psi''(t) = -c^2\lambda\psi(t)$$

does not have any boundary conditions, again as in the Dirichlet case, we have

$$\psi(t) = a\cos\left(\frac{n\pi}{l}t\right) + b\sin\left(\frac{n\pi}{l}t\right)$$

for some real a, b if $n > 0$ and, in this case,

$$\psi(t) = at + b$$

if $n = 0$.

We are still trying to solve the problem (2.12):

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on} \quad (0,l)\times(0,T) \\ \frac{\partial}{\partial r^1}(u)(0,t) = 0 = \frac{\partial}{\partial r^1}(u)(l,t) & \text{on} \quad [0,T] \\ u(x,0) = g(x) & \text{on} \quad [0,l] \\ u_t(x,0) = h(x) & \text{on} \quad [0,l]. \end{cases}$$

We again follow Fourier and form the cosine series

$$u(x,t) = \frac{1}{2}a_0 + \frac{1}{2}b_0 t + \sum_{n=1}^{\infty}\left(a_n\cos\left(c\frac{n\pi}{l}t\right) + b_n\sin\left(c\frac{n\pi}{l}t\right)\right)\cos\left(\frac{n\pi}{l}x\right).$$

Then u is a solution of (2.12) provided

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}x\right)$$

and

$$h(x) = \frac{1}{2}b_0 + c\sum_{n=1}^{\infty} \frac{n\pi}{l} b_n \cos\left(\frac{n\pi}{l}x\right)$$

If the boundary conditions are mixed, say

$$u(0,t) = \frac{\partial}{\partial r^1}(u)(l,t) = 0,$$

then

$$\lambda = \frac{\left(n+\frac{1}{2}\right)^2 \pi^2}{l^2}$$

and a corresponding eigenvector is

$$\sin \frac{\left(n+\frac{1}{2}\right)\pi}{l}.$$

This is part of what you want to show in your second homework assignment.

Similarly, the solution of

$$\begin{cases} u_t = ku_{xx} & \text{on} \quad (0,l)\times(0,T) \\ \frac{\partial}{\partial r^1}(u)(0,t) = 0 = \frac{\partial}{\partial r^1}(u)(l,t) & \text{on} \quad [0,T] \\ u(x,0) = g(x) & \text{on} \quad [0,l] \end{cases} \tag{2.14}$$

is

$$u = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-k(n\pi/l)^2 t} \cos\left(\frac{n\pi}{l}x\right),$$

provided $g(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$, and u and g are nice enough.

Here're some problems from your next homework. c.f. [Strauss, '92 pg 108].

1. Use the fact

$$\int_0^l \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

   to find the Fourier sine series of r^2 on $[0,l]$. That is, find a_n for which

$$r^2 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{l}x\right)$$

   on $[0,l]$.

2. Use a similar fact to find the Fourier cosine series of r^2 on $[0,l]$.

2.3 Periodic Boundary Conditions

We may also consider **periodic boundary conditions**. Here's another problem from your next homework.

From [Strauss, '92 pg 27]. **4.2.3.** Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \qquad \text{for } -l \le x \le l$$

$$u(-l,t) = u(l,t) \qquad \text{and} \qquad \frac{\partial}{\partial r^1}(u)(-l,t) = \frac{\partial}{\partial r^1}(u)(l,t).$$

These are called **periodic boundary conditions**.

1. Show the eigenvalues are

$$\lambda = \left(\frac{n\pi}{l}\right)^2$$

   for $n = 0, 1, 2, 3, \ldots$

2. Show the concentration is

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right)e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

Example 2.15. For example, suppose we want to find u for which

$$u_t = ku_{xx} \qquad \text{for } -l \le x \le l$$

$$u(-l,t) = u(l,t), \qquad \frac{\partial}{\partial r^1}(u)(-l,t) = \frac{\partial}{\partial r^1}(u)(l,t)$$

with

$$f(x) = u(x,0) = 1 + \cos\left(\frac{\pi}{l}x\right) + \sin\left(\frac{\pi}{2l}x\right).$$

If we can find sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right) \tag{2.16}$$

and if we define

$$u = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right)e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

then u is a solution of the problem in example 2.15.

Orthogonality of sin and cos We again make some observations. Let m and n be integers. Then

$$\int_{-l}^{l} \sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{m\pi}{l}x\right) = 0.$$

In particular,

$$\int_{-l}^{l} \sin\left(\frac{n\pi}{l}x\right) dx = 0.$$

And

$$\int_{-l}^{l} \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{m\pi}{l}x\right) dx = \begin{cases} l & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-l}^{l} \cos\left(\frac{n\pi}{l}x\right) \cos\left(\frac{m\pi}{l}x\right) dx = \begin{cases} l & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m. \end{cases}$$

Also,

$$\int_{-l}^{l} 1^2 dx = 2l.$$

To solve this problem with periodic boundary, we assume we can integrate term-by-term.

We notice, since $\sin\left(\frac{\pi}{2l}x\right)$ is an odd function in x and $\cos\left(\frac{n\pi}{l}x\right)$ is even in x for any integer n,

$$\int_{-l}^{l} \sin\left(\frac{\pi}{2l}x\right) \cos\left(\frac{m\pi}{l}x\right) dx = 0.$$

If we integrate (2.16) from $-l$ to l, then we notice $a_0 = 2$, since

$$2l = \int_{-l}^{l} dx = \int_{-l}^{l} f(x)dx = \int_{-l}^{l} \frac{1}{2}a_0 dx = a_0 l$$

If we multiply (2.16) by $\cos\left(\frac{n\pi}{l}x\right)$ and integrate from $-l$ to l, we obtain

$$a_1 = 1$$

and

$$a_n = 0$$

if $n > 1$, since

$$l = \int_{-l}^{l} f(x) \cos\left(\frac{\pi}{l}x\right) dx = a_1 l$$

and

$$0 = \int_{-l}^{l} f(x) \cos\left(\frac{n\pi}{l}x\right) dx = a_n l$$

if $n > 1$.

Finally, we note

$$\int_{-l}^{l} \sin\left(\frac{\pi}{2l}x\right) \sin\left(\frac{n\pi}{l}x\right) dx = \frac{2l}{\pi}(-1)^{n+1}\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)$$

after use of the identity $2\sin(\theta)\sin(\varphi) = \cos(\theta - \varphi) - \cos(\theta + \varphi)$ (product-to-sum). Hence,

$$\int_{-l}^{l} f(x) \sin\left(\frac{n\pi}{l}x\right) dx = \frac{2l}{\pi}(-1)^{n+1}\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)$$

This implies

$$b_n = \frac{2}{\pi}(-1)^{n+1}\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right).$$

Then

$$u = 1 + \left(\cos\left(\frac{\pi}{l}x\right) + \frac{8}{3\pi}\sin\left(\frac{\pi}{l}x\right)\right)e^{-k\left(\frac{\pi}{l}\right)^2 t}$$
$$-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)\sin\left(\frac{n\pi}{l}x\right)e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

on $(0, l) \times (0, T)$.

3 Fourier Series

Fourier Series

In the previous example we discovered the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx \tag{3.1}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx \tag{3.2}$$

provided $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{l}x\right) + b_n\sin\left(\frac{n\pi}{l}x\right)\right)$. Refer to [Strauss, '92 pg 103-107].

The following is from [Strauss, '92 pg 112-113]. Recall in Exercise 1 you are asked to show if

$$\begin{cases} \phi''(x) &= -\lambda\phi(x) \\ \phi(-l) &= \phi(l) \\ \phi'(-l) &= \phi'(l), \end{cases} \tag{3.3}$$

then

$$\lambda = \left(\frac{n\pi}{l}\right)^2$$

for some integer n. Then in Exercise 2 you are asked to show a basis of the vector space (over $\mathbb{R}$) of solutions of (3.3) is

$$\left\{\cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right)\right\}$$

if $\lambda = \left(\frac{n\pi}{l}\right)^2 \neq 0$ and

$$\left\{\frac{1}{2}\right\}$$

if $\lambda = 0$.

The hope is we can write many functions as series of such functions, as in the above example with f. Now suppose f is any real-valued function defined on $(-l, l)$, By using Euler's formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

if $n \in \mathbb{Z}$, then $e^{in\pi x/l}$ is an (complex) eigenvector (eigenfunction) of $-d^2/dr^2$ with eigenvalue $\left(\frac{n\pi}{l}\right)^2$. If $n \neq 0$, $e^{i\theta}$ and $e^{-i\theta}$ are linearly independent over $\mathbb{R}$. Therefore, we may write

$$f(x) = \sum_{n=-\infty}^{-1} c_n e^{in\pi x/l} + \sum_{n=0}^{\infty} c_n e^{in\pi x/l} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

formally for some c_n.

Notice if $n \neq m$ (in particular if they have different signs), then

$$\begin{aligned}\int_{-l}^{l} e^{in\pi x/l} e^{-im\pi x/l} dx \\ &= \int_{-l}^{l} e^{i\pi x(n-m)/l} dx \\ &= \frac{l}{i\pi(n-m)} \left(e^{i(n-m)\pi} - e^{i(m-n)\pi}\right) \\ &= \frac{l}{i\pi(n-m)} ((-1)^{n-m} - (-1)^{m-n}) \\ &= 0\end{aligned}$$

since $\cos((n-m)\pi) = (-1)^{n-m}$ and $(-1)^{n-m} = (-1)^{m-n}$.

If $n = m$, then

$$\int_{-l}^{l} e^{in\pi x/l} e^{-in\pi x/l} dx = \int_{-l}^{l} dx = 2l.$$

Finally, if

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l},$$

then

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in\pi x/l} dx.$$

The right hand side of $f(x)$ above is called the full Fourier series of f on $(-l, l)$ with complex coefficients c_n above.

In your next homework, you're asked to find the full Fourier series of e^x, $\cosh x$, $\sin x$ and $|x|$. These are from [Strauss, '92 pg 114]. Also from the same page, you're asked to show f is real-valued if and only if its complex Fourier coefficients satisfy $c_n = \overline{c_{-n}}$. In fact, here's a conjecture. Solutions of (3.3) are real if its boundary conditions are real.

Here's an example.

Example 3.4. Let's compute the full Fourier series of $f(x) = x$ on $(-1, 1)$. We have

$$c_n = \frac{1}{2} \int_{-1}^{1} x e^{-in\pi x} dx$$

if $n \neq 0$. Integration by parts yields

$$c_n = \frac{(-1)^{n+1}}{in\pi}.$$

Then $c_n e^{in\pi x} + c_{-n} e^{-in\pi x} = \frac{2(-1)^n}{n\pi}\sin(n\pi x)$. If $n = 0$, then $c_n = 0$. Hence, the full Fourier series of x on $(-1, 1)$ is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

If it's true that

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x),$$

then

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Notice if $n \neq m$ (in particular if they have different signs), then

$$\begin{aligned}\int_{-l}^{l} e^{in\pi x/l}e^{-im\pi x/l}dx \\ &= \int_{-l}^{l} e^{i\pi x(n-m)/l}dx \\ &= \frac{l}{i\pi(n-m)}\left(e^{i(n-m)\pi} - e^{i(m-n)\pi}\right) \\ &= \frac{l}{i\pi(n-m)}((-1)^{n-m} - (-1)^{m-n}) \\ &= 0\end{aligned}$$

since $\cos((n-m)\pi) = (-1)^{n-m}$ and $(-1)^{n-m} = (-1)^{m-n}$.

If $n = m$, then

$$\int_{-l}^{l} e^{in\pi x/l}e^{-in\pi x/l}dx = \int_{-l}^{l} dx = 2l.$$

Finally, if

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l},$$

then

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx.$$

The right hand side of $f(x)$ above is called the full Fourier series of f on $(-l, l)$ with complex coefficients c_n above.

Here's an example.

Example 3.5. Let's compute the full Fourier series of $f(x) = x$ on $(-1, 1)$. We have

$$c_n = \frac{1}{2}\int_{-1}^{1} xe^{-in\pi x}dx$$

if $n \neq 0$. Integration by parts yields

$$c_n = \frac{(-1)^{n+1}}{in\pi}.$$

Then $c_n e^{in\pi x} + c_{-n} e^{-in\pi x} = \frac{2(-1)^n}{n\pi} \sin(n\pi x)$. If $n = 0$, then $c_n = 0$. Hence, the full Fourier series of x on $(-1, 1)$ is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x).$$

If it's true that

$$x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)$$

point-wise on $(-1, 1)$, then

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

3.1 General Fourier Series and square integrable functions

General Fourier Series and square integrable functions From [Strauss, '92 pg 114-118]. We now consider the closed interval $[a, b]$ if $a < b$. We'll write $\int_a^b f dx$ instead of $\int_a^b f(x)dx$ whenever the context is clear. We say a complex-valued function f is **square-integrable** on $[a, b]$ if

$$\int_a^b |f(x)|^2 dx < \infty.$$

Here $|f(x)|^2 = f(x) \cdot \overline{f(x)}$ denotes the norm of $f(x)$ as a complex number, and the line over $f(x)$ means the complex conjugate of $f(x)$: $\overline{z} = re^{-i\theta}$ if $z = re^{i\theta}$ and $r, \theta \in \mathbb{R}$.

- We denote the space of complex-valued square-integrable functions on $[a, b]$ by

  $$L^2[a, b].$$

- $L^2[a, b]$ is a vector space over $\mathbb{C}$ with pointwise addition and scalar multiplication.

- We'll also refer to $C^2(a, b)$, the space of complex-valued functions whose second derivative is continuous on (a, b).

- The map, B, sending f and g in $L^2[a, b]$ to

  $$B(f, g) = \int_a^b f(x)\overline{g(x)} dx$$

  sends $L^2[a, b] \times L^2[a, b]$ into $\mathbb{R}$.

- The above map is a complex inner product, different from the real dot product defined in lecture 1.

- Then there's a norm induced by the inner product above given by

  $$\|f\| = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}} = \sqrt{B(f, f)}$$

  for all $f \in L^2[a, b]$.

- Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $L^2[a,b]$, we say

$$f_n \to f \text{ in } L^2[a,b] \text{ as } n \to \infty$$

(f_n converges to f in the L^2 norm) if the sequence of real numbers $\|f_n - f\| \to 0$ as $n \to \infty$.

As an aside, the Lebesgue integral rather than the Riemann integral gives us the most generality. Also, B technically isn't an inner product unless we consider either continuous functions or equivalence classes of functions which agree almost everywhere, which is what $L^2[a,b]$ is usually defined to be anyway.

Consider the linear operator $A = -d^2/dr^2$ acting on complex-valued functions defined on (a,b) with continuous second derivatives there. Unless otherwise stated, any eigenvector, φ, of A satisfies either the Dirichlet, Nuemann, or periodic boundary conditions on (a,b).

1. **Dirichlet:** $\varphi(a) = 0 = \varphi(b)$.
2. **Nuemann:** $\varphi'(a) = 0 = \varphi'(b)$.
3. **periodic:** $\varphi(a) = \varphi(b)$ and $\varphi'(a) = \varphi'(b)$.

Suppose φ, υ are eigenvectors of A with corresponding eigenvalues λ, μ, respectively. Integration by parts and the boundary conditions yield

$$B(A\,\varphi, \upsilon) - B(\varphi, A\,\upsilon) = 0.$$

Then

$$(\lambda - \mu)B(\varphi, \upsilon) = 0.$$

So, if λ and μ are distinct eigenvalues, then φ and υ are orthogonal.

If two eigenvectors with the same eigenvalue are not orthogonal, we can make them orthogonal via the Gram-Schmidt process.

Theorem 3.6. 1. The eigenvalues of A are nonnegative and form an increasing unbounded sequence.

2. If $f \in L^2[a,b]$, φ_n are all the eigenvectors of A and are pairwise orthogonal, $c_n = \frac{B(f,\varphi_n)}{\|\varphi_n\|^2}$ and $S_N = \sum_{n=1}^{N} c_n\,\varphi_n$, then

$$S_N \to f \text{ in } L^2[a,b] \text{ as } N \to \infty.$$

We'll try to prove Theorem 3.6 when

- $b \neq 0$ and $a = -b = -l$
- $f \in C^2(-l,l)$, $f(-l) = f(l)$ and $f'(-l) = f(l)$.
- $\varphi_n(x) = e^{in\pi x/l}$ on $(-l,l)$ for any integer n.

1. This is one of your homework problems and we note again the eigenvalue corresponding to φ_n and φ_{-n} is

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2.$$

2. We denote the remainder by

$$r_N = f - S_N.$$

It can be shown

$$\lambda_N \leq \frac{\|r_N'\|^2}{\|r_N\|^2}$$

[Strauss, '92 pg 286 and 288]. We also have, from integration by parts, the boundary conditions and orthogonality,

$$\|r_N'\|^2 = \|f'\|^2 - \sum_{n=1}^{N} \overline{c_n} B(f', \varphi_n') - \sum_{n=1}^{N} c_n B(\varphi_n', f') + \sum_{n=1}^{N} c_n \overline{c_m} B(\varphi_n', \varphi_m')$$

$$= \|f'\| - \sum_{n=1}^{N} \overline{\lambda_n} \overline{c_n} B(f, \varphi_n) - \sum_{n=1}^{N} \lambda_n c_n B(\varphi_n, f) + \sum_{n=1}^{N} \lambda_n |c_n|^2 \| \varphi_n \|^2.$$

Now, since $\overline{\lambda_n} = \lambda_n$, $c_n = \frac{B(f,\varphi_n)}{\| \varphi_n \|^2}$ and $B(\varphi_n, f) = \overline{B(f, \varphi_n)}$, it follows

$$\|r_N'\|^2 = \|f'\|^2 - \sum_{n=1}^{N} \lambda_n |c_n|^2 \| \varphi_n \|^2.$$

This, together with

$$\lambda_N \leq \frac{\|r_N'\|^2}{\|r_N\|^2},$$

implies

$$\|r_N\|^2 \leq \frac{\|f'\|^2}{\lambda_N}.$$

From 1, we have the result. This is from [Strauss, '92 pg 296].

$$\lambda_N \leq \frac{\|r_N'\|^2}{\|r_N\|^2}$$

follows from the facts

$$\lambda_N = \min\{ \frac{\|w'\|}{\|w\|} \mid w \not\equiv 0, w(l) = w(-l),$$

$$w'(l) = w'(-l), w \in C^2(-l, l), B(w, \varphi_n) = 0 \text{ if } n < N \}$$

and

$$B(r_N, \varphi_n) = 0 \text{ if } n < N.$$

We discuss further the L^2 theory of Fourier series. This is from [Strauss, '92 pg 126-129]. Recall $c_n = \frac{B(f,\varphi_n)}{\| \varphi_n \|^2}$ with $B(f, g) = \int_a^b f(x)\overline{g(x)}dx$ and $\|f\| = \sqrt{B(f, f)}$. Also, $S_N = \sum_{n=0}^{N} c_n \varphi_n$ and $r_N = f - S_N$. Then

$$0 \leq \|r_N\|^2 = \|f\|^2 - \sum_{n=0}^{N} c_n^2 \| \varphi_n \|^2$$

in the same way we computed $\|r'_N\|^2$. Then

$$\sum_{n=0}^{N} c_n^2 \| \varphi_n \|^2 \leq \|f\|^2.$$

The left hand side of the last equation is an increasing sequence in N and is bounded if $f \in L^2[a,b]$. Hence, if $f \in L^2[a,b]$, we have **Bessel's inequality:**

$$\sum_{n=0}^{\infty} c_n^2 \int_a^b |\varphi_n(x)|^2 dx \leq \int_a^b |f(x)|^2 dx.$$

Then since S_N converges to f in L^2 if and only if $\|r_N\|^2 \to 0$ as $N \to 0$, it follows S_N converges to f in L^2 if and only if

$$\sum_{n=0}^{\infty} c_n^2 \int_a^b |\varphi_n(x)|^2 dx = \int_a^b |f(x)|^2 dx.$$

This equation is know as **Parseval's equality**. From the theorem we attempted to prove last time, Parseval's equality is true for all $f \in L^2[a,b]$, as long as the eigenvectors are from the Dirichlet, Neumann or periodic boundary value problems.

As an example, the Dirichlet problem on $(0,\pi)$ has eigenvectors $\sin(nx)$ and the Fourier series of 1 on this interval is

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nx).$$

Certainly $1 \in L^2[0,\pi]$, so that Parseval's equality asserts

$$\sum_{n \text{ odd}} \left(\frac{4}{n\pi}\right)^2 \int_0^\pi \sin^2(nx) dx = \int_0^\pi 1^2 dx.$$

Then

$$\sum_{n \text{ odd}} \left(\frac{4}{n\pi}\right)^2 \frac{\pi}{2} = \pi.$$

Hence,

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Recall we said $f_n \to f$ in $L^2[a,b]$ if $\|f_n - f\| \to 0$ as $n \to \infty$, with $\|f_n - f\|^2 = \int_a^b |f_n(x) - f(x)|^2 dx$.

Other forms of convergence. From [Struass, '92 pg 120-125]

We now discuss other forms of convergence. We say a sequence of functions f_n, defined on (a,b) converges **pointwise** to f on (a,b) if, for each $x \in (a,b)$,

$$\left|f(x) - f_n(x)\right| \to 0 \text{ as } n \to \infty.$$

We say f_n converges **uniformly** to f on $[a,b]$ if

$$\max_{a \leq x \leq b} \left|f(x) - f_n(x)\right| \to 0 \text{ as } n \to \infty.$$

It should be clear that uniform convergence implies pointwise convergence. In your homework, you're asked to show uniform convergence implies L^2 convergence. Thus, uniform convergence is stronger than pointwise and L^2 convergence. Below we show it's strictly stronger than both notions of convergence.

For example, if $f_n(x) = 1 - x^n$ on $[0, 1]$, then f_n converges pointwise to $f(x) = 1$ on $(0, 1)$. However, it does not converge uniformly to f on $[0, 1]$, since $|f(x) - f_n(x)| = x^n$ and $|f(1) - f_n(1)| = 1$.

But $\int_0^1 x^{2n} dx = \frac{1}{2n+1}$, so that $f_n \to f$ in $L^2[0, 1]$.

These next two examples are from https://web.math.ucsb.edu/ grigoryan/124B/lecs/lec5.pdf

For example, if $f_n(x) = n$ if $0 \le x < \frac{1}{n}$ and $f_n(x) = 0$ if $\frac{1}{n} \le x \le 1$ and $f_0(x) = 0$ on $[0, 1]$, then $f_n \to 0$ pointwise on $(0, 1)$.

However, $\int_0^1 (f_n(x))^2 dx = n^2 \to \infty$ as $n \to \infty$. So, f_n does not converge to 0 in $L^2[0, 1]$.

For example, if $f_n(x) = (-1)^n$ if $x = \frac{1}{2}$ and $f_n(x) = 0$ if $x \in [0, 1] \setminus \{\frac{1}{2}\}$, with $f_0(x) = 0$ on $[0, 1]$, then f_n does not convergence pointwise to any function on $(0, 1)$, since $f_n(\frac{1}{2})$ does not converge to any number.

However, $\int_0^1 (f_n(x))^2 dx = 0$ for all n, so that $f_n \to 0$ in $L^2[0, 1]$.

These two examples show L^2 and pointwise convergence are neither stronger nor weaker than the other. However, it can be shown if f_n are continuous and $f_n \to f$ in L^2, then $f_n \to f$ pointwise.

Now we state some convergence theorems. Again A is the operator $-d^2/dr^2$ on the problem space of Dirichlet, Neumann, or periodic boundary conditions on $[a, b]$. We've established its eigenvalues there form a nonnegative, increasing, unbounded sequence, λ_n. If φ_n is an eigenvector associated to λ_n, f is a function on $[a, b]$ and $c_n = \frac{B(f, \varphi_n)}{\| \varphi_n \|^2}$, then $\sum_{n=0}^{\infty} c_n \varphi_n$ is called the **Fourier series** of f on $[a, b]$. Set $S_N = \sum_{n=0}^{N} c_n \varphi_n$.

We've attempted to prove

Theorem 3.7. If $f \in L^2[a, b]$, then S_N converges to f in $L^2[a, b]$.

As a remark, our 'proof' relied on the assumption $f \in C^2(a, b)$. In fact, it can be shown $C^2(a, b)$ is **dense** in $L^2[a, b]$, meaning every function in $L^2[a, b]$ has a sequence in $C^2(a, b)$ which converges to it in $L^2[a, b]$.

We state the pointwise convergence theorem, whose proof can be found in [Struass, '92 pg 132-135]. A function, f, is called **piecewise continuous** on $[a, b]$ if its one-sided limits, $f(x+)$ and $f(x-)$ exist at every point $x \in [a, b]$.

Theorem 3.8. If f and f' are piecewise continuous on $[a, b]$, then $S_N(x)$ converges to $\frac{1}{2}(f(x+) + f(x-))$ for all $x \in (a, b)$.

The endpoints are a more delicate matter. For example, on $[-l, l]$, consider the **periodic extension** of f: $f_{per}(x) = f(x - 2lm)$ on $(-l + 2lm, l + 2lm)$ for all integers m. In general, f_{per} is discontinuous at the endpoints of $(-l+2lm, l+2lm)$. Then Fourier's Theorem ([Haberman, '08 pg 92]) states (with the conditions in the theorem), the Fourier series converges pointwise to

$$\frac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)$$

on $[-l, l]$.

We also list

Theorem 3.9. If $f \in C^2(a,b)$, f, f', f'' are continuous on $[a,b]$, and f satisfies the boundary conditions that the φ_n do, then S_N converges uniformly to f on $[a,b]$.

A proof of this theorem can be found in [Strauss, '92 pg 135-136].

To finish our discussion of Fourier series, we list some theorems discussing the term-by-term differentiation and integration of them.

Theorem 3.10. From [Haberman, '08 pg 118]. If $f(-l) = f(l)$, f is continuous, and f' is piecewise continuous, then the Fourier series can be differentiated term-by-term. The resultant series is the Fourier series of f'.

Here's a nonexample. Recall the Fourier series of x on $(-1,1)$ is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

From the pointwise convergence theorem, we know

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$$

if $x \in (-1,1)$. Let's differentiate the right hand side term-by-term:

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\cos(n\pi x).$$

This is certainly not 1 since the Fourier series of 1 on $(-1,1)$ is just 1 itself.

Theorem 3.11. From [Haberman, '08 pg 127]. If f and f' are piecewise continuous on $[-l,l]$, then the Fourier series can be integrated term-by-term. The resultant series converges to the integral of f on $[-l,l]$.

The proof can be found in [Haberman, '08 pg 129-130].

4 Laplace's Equation

From Strauss section 6.1. Instead of studying the wave and heat equation in higher spatial dimensions, we first study the stationary case in higher spatial dimensions: Laplace's and Poisson's equation. The stationary case in one spatial dimension is boring and we've solved it extensively: if $u_t = 0$ or $u_{tt} = 0$, then in one spatial dimension, $u_{xx} = 0$.

We've mentioned Laplace's equation in two dimensions:

$$u_{xx} + u_{yy} = 0.$$

In three dimensions, it's

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

If you wish, the Laplace operator, sending u to $u_{xx} + u_{yy} + u_{zz}$ is the trace of the Hessian operator, which is the matrix of second partials. We'll use the notation $\Delta u = u_{xx} + u_{yy} + u_{zz}$ or $\Delta u = u_{xx} + u_{yy}$. The context will be clear which dimension we are considering. The operator Δ is called the **Laplacian** or Laplace operator. Other than being the trace of the Hessian, it's also the divergence of the gradient. Recall the gradient of u is the row vector $\nabla u = (\frac{\partial}{\partial r^1}(u), \frac{\partial}{\partial r^2}(u))$. The **divergence** of a vector field (a,b) is defined as the function $\nabla \cdot (a,b) = a_x + b_y$.

- A solution to Laplace's equation $\Delta u = 0$ is called **harmonic**.
- Harmonic functions are stationary solutions of the wave and heat equations.
- The real and imaginary parts of complex analytic functions are necessarily harmonic, as shown from the Cauchy-Riemann equations.

By the Laplacian in three dimensions, we mean the operator on real-valued functions defined on a subset of $\mathbb{R}^3$, the set of all ordered triples of real numbers. We describe the canonical basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in a similar way with coordinates as in $\mathbb{R}^n$, along with the dot product, the norm, the coordinate functions, open disks, open sets, limits, continuity, and partial derivatives. In particular, an open set, $D \subseteq \mathbb{R}^3$ is a set which is the union of open disks

$$B_r(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^3 \mid |\mathbf{y} - \mathbf{z}| < r\}$$

and its boundary, bdy D, is the set of all points such that every open disk centered at a point there contains points in D and points in $\mathbb{R}^3 \setminus D$.

- **Poisson's equation** is just the **inhomogeneous** version of Laplace's equation: $\Delta u = f$ with f a prescribed function.
- We can ask about solutions of either of the boundary value problems

$$\begin{aligned} \Delta u &= f \qquad \text{on } D \\ u &= h \qquad \text{on bdy } D \end{aligned}$$

or

$$\begin{aligned} \Delta u &= f \qquad \text{on } D \\ \frac{\partial}{\partial u}\mathbf{n} &= h \qquad \text{on bdy } D, \end{aligned}$$

Again called the Dirichlet or Neumann problems.

Here, $\frac{\partial}{\partial u}\mathbf{n}$ is a certain directional derivative of u: $\frac{\partial}{\partial u}\mathbf{n} = \nabla u \cdot \mathbf{n}$, with $\mathbf{n}$ a choice of unit normal of bdy D. We just remark here if bdy D is described by say a level set $\gamma(\mathbf{z}) = 0$ with $\nabla\gamma$ never zero, then a choice of unit normal is

$$\mathbf{n} = \frac{\nabla\gamma}{|\nabla\gamma|}.$$

Recall and compare the statement of the existence and uniqueness of solutions of quasilinear first order pde in two variables. Somewhat unlike that result, the geometry of D (of bdy D) greatly affects the solvability of Poisson's boundary value problems. It becomes difficult to say anything in full generality.

Theorem 4.1. The Weak Maximum Principle. Let D be a bounded open set. Let u be a harmonic function in D which is continuous on $\overline{D} = D \cup \text{bdy } D$. Then the maximum and minimum of u on $\overline{D}$ occur on bdy D.

Proof:

This result is related to the second derivative test. If a maximum of u occurs in D, then $\Delta u \leq 0$ there. Suppose u is harmonic in D. Fix $\varepsilon > 0$. If we define $v(\mathbf{z}) = u(\mathbf{z}) + \varepsilon\,|\mathbf{z}|^2$ on $\overline{D}$, then, since $\Delta|\mathbf{z}|^2$ is constant and positive,

$$\Delta v = 0 + \varepsilon\,\Delta|\mathbf{z}|^2 > 0$$

on D. Therefore, v does not attain a local maximum in D.

It can be proved $\overline{D}$ is closed, meaning it's the complement of an open set, and continuous functions on a closed and bounded set attain their global extrema there (extreme value theorem). Since u is continuous on $\overline{D}$, so is v, so its global maximum occurs at some point $\mathbf{z}_0 \in$ bdy D. Since D is bounded, there is some $R > 0$ such that $\overline{D} \subseteq B_R(0)$, so that, for all $\mathbf{z} \in D$,

$$u(\mathbf{z}) \leq v(\mathbf{z}) \leq v(\mathbf{z}_0) = u(\mathbf{z}_0) + \varepsilon\,|\mathbf{z}_0|^2 \leq \max_{\text{bdy}\,D} u + \varepsilon\,R^2.$$

Since $\varepsilon > 0$ is arbitrary,

$$u(\mathbf{z}) \leq \max_{\text{bdy}\,D} u$$

for all $\mathbf{z} \in D$.

If we consider $u(\mathbf{z}) - \varepsilon\,|\mathbf{z}|^2$ then we obtain a similar result about the minimum of u.

The weak maximum principle is strong enough to allow us to infer the uniqueness of Poisson's Dirichlet problem.

Theorem 4.2. If D is open, and bounded, f and h are prescribed functions, then there is at most one solution, u, of

$$\begin{aligned} \Delta u &= f \qquad \text{on } D \\ u &= h \qquad \text{on bdy } D, \end{aligned}$$

continuous on $\overline{D}$.

Proof: If u and v are two such solutions of the above problem and $w = u - v$, then $\Delta w = 0$ (w is harmonic), $w = 0$ on bdy D, and w is continuous on $\overline{D}$. By the weak maximum principle, for all $\mathbf{z} \in \overline{D}$,

$$0 = \min_{\text{bdy}\,D} w \leq w(\mathbf{z}) \leq \max_{\text{bdy}\,D} w = 0.$$

4.1 Fundamental solution

It can be shown, if T is an affine translation of $\mathbb{R}^n$: $T(x) = x + y$, or R is a rotation: R is linear and $R^T R = I$, with I the identity matrix and R^T the transpose of R, then the Laplacian is invariant under T and R. Meaning $\Delta(u \circ R) = \Delta u \circ R$. In the plane, in $\mathbb{R}^n$, if $x = r\cos\theta$ and $y = r\sin\theta$, then by the chain rule

$$\Delta = \frac{\partial}{\partial^2} r^2 + \frac{1}{r}\frac{\partial}{\partial} r + \frac{1}{r^2}\frac{\partial}{\partial^2}\theta^2.$$

We search for **radial** harmonic functions: $u = u \circ R$ for all rotations R. Since if u is radial, then $\Delta(u \circ R) = \Delta u$. If u is radial in the plane, then $u(r, \theta) = u(r)$, so that if u is harmonic and radial, then

$$\frac{\partial}{\partial^2 u} r^2 + \frac{1}{r}\frac{\partial}{\partial u} r = 0.$$

We can integrate

$$u''/u' = -1/r$$

twice or notice $(r u_r)_r = 0$. This has the solution

$$u = a + b\log(r)$$

with $a, b \in \mathbb{R}$ and is called the **fundamental solution** of the Laplacian in $\mathbb{R}^n$. $a + b\log(r)$ is harmonic on $\mathbb{R}^n \setminus \{0\}$.

In space, in $\mathbb{R}^3$, if

$$x = r\sin\phi\cos\theta,$$

$$y = r\sin\phi\sin\theta$$

and

$$z = r\cos\phi,$$

(spherical coordinates) then

$$\Delta = \frac{\partial}{\partial^2}r^2 + \frac{2}{r}\frac{\partial}{\partial}r + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\sin\phi\frac{\partial}{\partial}\phi}\phi + \frac{1}{r^2\sin^2\phi}\frac{\partial}{\partial^2}\theta^2.$$

Again, if u is harmonic and radial, then

$$\frac{\partial}{\partial^2 u}r^2 + \frac{2}{r}\frac{\partial}{\partial u}r = 0.$$

Then we may integrate

$$u''/u' = \frac{2}{r}$$

twice or notice $(r^2u_r)_r = 0$ to obtain

$$u = ar^{-1} + b$$

called the fundamental solution of the Laplacian in $\mathbb{R}^3$.

4.2 Poisson's Formula

Poisson's Formula From [Strauss, '92 pg 159-163 section 6.3].

We now specialise and consider the Dirichlet problem of Laplace's equation in a *disk* in $\mathbb{R}^n$. In particular, a disk of radius R centered at the origin.

$$\Delta u = 0 \qquad \text{on } B_R(0)$$

$$u = h \qquad \text{on bdy } B_R(0).$$

$B_R(0)$ can be described by the inequality

$$r^2 + y^2 < R^2$$

and its boundary by the level set

$$r^2 + y^2 = R^2.$$

Recall the Laplacian in polar coordinates is

$$\Delta = \frac{\partial}{\partial^2}r^2 + \frac{1}{r}\frac{\partial}{\partial}r + \frac{1}{r^2}\frac{\partial}{\partial^2}\theta^2.$$

We separate the polar coordinates of u: if $\Delta u = 0$ and $u = \varphi(r)\psi(\theta)$, then

$$\psi\,\varphi'' + \frac{1}{r}\cdot\varphi'\,\psi + \frac{1}{r^2}\cdot\varphi\,\psi'' = 0$$

for all $(r, \theta) \in (0, R) \times [0, 2\pi)$.

If we multiply this expression by $\frac{r^2}{\varphi \psi}$, we get

$$r^2 \cdot \frac{\varphi''}{\varphi} + r \cdot \frac{\varphi'}{\varphi} + \frac{\psi''}{\psi} = 0.$$

Then $r^2 \cdot \frac{\varphi''}{\varphi} + r \cdot \frac{\varphi'}{\varphi}$ and $-\frac{\psi''}{\psi}$ are both the same constant, λ, since they're equal and one depends only on r, the other only on θ.

We end up with the separated ODE's

$$r^2 \varphi'' + r \varphi' - \lambda \varphi = 0$$

and

$$\psi'' + \lambda \psi = 0.$$

Since ψ is the angular part of u, it's period is 2π:

$$\psi(\theta + 2\pi) = \psi(\theta)$$

for all real θ. Again, $\psi'' + \lambda\psi = 0$ implies

$$\psi(\theta) = a e^{\sqrt{-\lambda} \cdot \theta} + b e^{-\sqrt{-\lambda} \cdot \theta}$$

for some $a, b \in \mathbb{C}$. Then periodicity implies

$$a e^{\sqrt{-\lambda} \cdot 2\pi} + b e^{-\sqrt{-\lambda} \cdot 2\pi} = a + b$$

but also, if $\lambda \neq 0$,

$$a e^{\sqrt{-\lambda} \cdot 2\pi} - b e^{-\sqrt{-\lambda} \cdot 2\pi} = a - b$$

as the derivative of ψ is then also 2π-periodic.

If $a \neq 0$, then

$$e^{\sqrt{-\lambda} \cdot 2\pi} = 1$$

and hence

$$2\pi \sqrt{-\lambda} = 2\pi n i$$

for some nonzero integer n. If $a = 0$ and ψ is not zero, then $b \neq 0$ and similarly

$$-2\pi \sqrt{-\lambda} = 2\pi n i$$

for some nonzero integer n. In any case,

$$\lambda = n^2$$

for some nonzero integer n. If $\lambda = 0$ and ψ is not zero, then ψ is some nonzero constant.

The equation

$$r^2 \varphi'' + r \varphi' - n^2 \varphi = 0$$

is a Cauchy-Euler equation. If $\varphi(r) = r^\alpha$, then

$$r^\alpha (\alpha(\alpha - 1) + \alpha - n^2) = 0.$$

So,

$$\alpha = \pm n.$$

Then $\varphi(r) = cr^n + dr^{-n}$ for some constants c, d. If $n = 0$, then another solution of the above equation is $\log(r)$. But r^{-n} and $\log(r)$ are not finite at $r = 0$ when $n \neq 0$. This is our auxiliary condition of φ. If n is a nonnegative integer and if $c_n, c_{-n} \in \mathbb{C}$, then

$$r^n(c_n e^{in\theta} + c_{-n} e^{-in\theta})$$

is harmonic on $B_R(0)$, satisfies a periodic boundary condition and is finite at the origin.

Recall $u = h$ when $r^2+y^2 = R^2$. Noting $r^2 = r^2+y^2$, if h is say continuously differentiable and periodic, then it converges pointwise to its full Fourier series

$$h(\theta) = c_0 + \sum_{n=1}^{\infty} R^n(c_n e^{in\theta} + c_{-n} e^{-in\theta}),$$

with

$$c_n = \frac{1}{2\pi R^n} \int_0^{2\pi} h(\phi) e^{-in\phi} d\phi$$

for any integer n.

Hence,

$$u = \sum_{n=1}^{\infty} \frac{r^n}{2\pi R^n} \left(\int_0^{2\pi} h(\phi) e^{-in\phi} d\phi e^{in\theta} + \int_0^{2\pi} h(\phi) e^{in\phi} d\phi e^{-in\theta} \right)$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi$$

is harmonic on $B_R(0)$, is finite at the origin, and $u = h$ when $r = R$.

Now,

$$\sum_{n=1}^{\infty} \frac{r^n}{2\pi R^n} \left(\int_0^{2\pi} h(\phi) e^{-in\phi} d\phi e^{in\theta} + \int_0^{2\pi} h(\phi) e^{in\phi} d\phi e^{-in\theta} \right)$$
$$= \frac{1}{2\pi} \int_0^{2\pi} h(\phi) \sum_{n=1}^{\infty} \frac{r^n}{R^n} \left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)} \right) d\phi.$$

We notice

$$\sum_{n=1}^{\infty} \left(\frac{re^{i(\theta-\phi)}}{R} \right)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{re^{-i(\theta-\phi)}}{R} \right)^n$$

are each convergent geometric series in $B_R(0)$. The fact that these series are geometric also justifies why, by the comparison test and integral convergence theorem for series of functions (compare [Strauss, '92 pg 388 in the $r^k ix$]), we can commute the series and integral of the first computation in this slide.

These geometric series telescope to

$$\frac{re^{i(\theta-\phi)}}{R - re^{i(\theta-\phi)}} \quad \text{and} \quad \frac{re^{-i(\theta-\phi)}}{R - re^{-i(\theta-\phi)}}$$

respectively. And

$$\frac{re^{i(\theta-\phi)}}{R - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{R - re^{-i(\theta-\phi)}} = \frac{Rre^{i(\theta-\phi)} - r^2 + Rre^{-i(\theta-\phi)} - r^2}{R^2 - Rre^{-i(\theta-\phi)} - Rre^{i(\theta-\phi)} + r^2}$$

$$= \frac{2Rr\cos(\theta-\phi)-2r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}.$$

So,

$$1+\sum_{n=1}^{\infty}\frac{r^n}{R^n}\left(e^{in(\theta-\phi)}+e^{-in(\theta-\phi)}\right)=\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}.$$

Hence,

$$u(r,\theta)=\frac{(R^2-r^2)}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi.$$

This is **Poisson's Formula.**

If you wish, we may rewrite this as a line integral over the curve $C = \text{bdy}\, B_R(0) = \{r^2+y^2 = R^2\}$. Recall the line integral of a complex-valued function f over C is defined as

$$\int_C f(\mathbf{x})ds(\mathbf{x}) = \int_a^b f(\mathbf{x}(t))|\mathbf{x}'(t)|dt$$

if $\mathbf{x} : [a,b] \to C$ is any bijection. In our case, $\mathbf{x} : [0,2\pi) \to C$ is, say, $\mathbf{x}(\phi) = (R\cos(\phi), R\sin(\phi))$ if $\phi \in [0,2\pi)$. This is just the polar coordinate parametrization of the curve C. Then

$$|\mathbf{x}'(t)| = R \quad \text{and} \quad h(\phi) = u(\mathbf{x}(\phi)).$$

The law of cosines implies

$$|\mathbf{z}-\mathbf{x}(\phi)|^2 = R^2+r^2-2Rr\cos(\theta-\phi)$$

if $\mathbf{z} = (r,\theta)$ in polar coordinates.

Hence,

$$u(\mathbf{z}) = \frac{(R^2-|\mathbf{z}|^2)}{2\pi R}\int_0^{2\pi}\frac{Ru(\mathbf{x}(\phi))}{|\mathbf{z}-\mathbf{x}(\phi)|^2}d\phi$$

$$= \frac{(R^2-|\mathbf{z}|^2)}{2\pi R}\int_C\frac{u(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^2}ds(\mathbf{x}).$$

This immediately gives the **Mean Value Property:**

The value of a harmonic function at the center of a disk is equal to its average on the circumference of that disk.

Suppose u is harmonic on a disk $D = B_R(\mathbf{z}_0)$ and continuous on $\overline{D}$. Make a change of coordinates $T(\mathbf{z}) = \mathbf{z}+\mathbf{z}_0$ as a map from the plane to itself. Define $\tilde{u} = u \circ T$. Then $\tilde{u}$ is harmonic on $T^{-1}(D) = B_R(0)$ continuous on $\overline{B_R(0)}$, and Poisson's Formula gives

$$\tilde{u}(0) = \frac{R^2}{2\pi R}\int_{\text{bdy}\, B_R(0)}\frac{\tilde{u}(\mathbf{x})}{|\mathbf{x}|^2}ds(\mathbf{x}) = \frac{1}{2\pi R}\int_{\text{bdy}\, B_R(0)}\tilde{u}(\mathbf{x})ds(\mathbf{x}).$$

Finally,

$$u(\mathbf{z}_0) = \frac{1}{2\pi R}\int_{\text{bdy}\, D}u(\mathbf{x})ds(\mathbf{x})$$

by substitution.

The mean value property implies

Theorem 4.3. The Strong Maximum Principle

If u is harmonic on a connected, open, and bounded set D in the plane, and continuous on $\overline{D}$, then either u is constant on $\overline{D}$ or its maximum and minimum only occur on bdy D.

For a proof, since u is continuous on the closed and bounded set $\overline{D}$, the maximum of u, say M, occurs at some point in $\overline{D}$, say $\mathbf{z}_M$. Suppose in fact $\mathbf{z}_M \in D$. Since D is open, there is some disk $\overline{B_R(\mathbf{z}_M)} \subseteq D$. By the mean value property,

$$M = u(\mathbf{z}_M) = \frac{1}{2\pi R}\int_{\text{bdy } B_R(\mathbf{z}_M)} u \quad ds \leq M.$$

That is, the average of u on the circle bdy $B_R(\mathbf{z}_M)$ is M. Since this is the maximum of u, it follows $u(\mathbf{z}) = M$ for all $\mathbf{z} \in$ bdy $B_R(\mathbf{z}_M)$.

We can apply the mean value property for any $r < R$. Therefore, $u(\mathbf{z}) = M$ for all $\mathbf{z} \in \overline{B_R(\mathbf{z}_M)}$.

Since D is open and connected in $\mathbb{R}^n$, it's path-connected, meaning every two points in D are the endpoints of a curve contained entirely in D. Consider any other point in D, say $\mathbf{z}_0$. There exists a curve from $\mathbf{z}_M$ to $\mathbf{z}_0$ contained in D. Cover the curve with overlapping disks whose closures are contained in D. Then we can apply the mean value property along the curve to conclude $u(\mathbf{z}_0) = M$.

That is, if $u = M$ somewhere in D, then $u = M$ on D. Otherwise, of course, if the maximum nor the minimum isn't attained in D, then continuity implies the extrema occur on the boundary.

■

If u is harmonic on an open set D in the plane, then u is **smooth** on D. That is, every partial derivative of any order of u exists and is continuous on D.

If $\mathbf{z}_0 \in D$, then after a suitable translation, we may assume $\mathbf{z}_0$ is contained in a disk centered at the origin, with say radius R, whose closure is contained in D. Hence,

$$u(\mathbf{z}) = \frac{(R^2 - |\mathbf{z}|^2)}{2\pi R}\int_C \frac{u(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^2} ds(\mathbf{x})$$

for all $\mathbf{z}$ in this disk. Certainly $\frac{u(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^2}$ is differentiable with respect to $\mathbf{z}$ to all orders away from the boundary, C. So, at, say, $\mathbf{z}_0$, we may differentiate under the integral sign and conclude u is smooth at $\mathbf{z}_0$. Since translations are smooth, and compositions of smooth functions are smooth, the original function is smooth at $\mathbf{z}_0$. Since $\mathbf{z}_0 \in D$ is arbitrary, we conclude u is smooth on D.

■

If you wish, please read and follow chapter 6.4 in Strauss's text called Circles, Wedges, and Annuli. These special geometries determine the boundary conditions for Laplace's equation and hence the behavior of the harmonic functions found there. Refer if you wish to Viktor Grigoryan's lecture notes on the same topic, posted on Canvas in this same module.

5 Green's Functions

Green's Functions For the remainder of this week and next week, we'll be studying Green's functions, the existence of which for a given domain is equivalent to the existence of a harmonic function there. Content is found in [Strauss, '92 ch 7.1-7.3].

First, we introduce some notation and some preliminary results. Think of working in $\mathbb{R}^3$, but really these computations will work in $\mathbb{R}^n$. A real-valued function whose partials are continuous on a open subset of $\mathbb{R}^3$ has gradient

$$\operatorname{grad} f = \nabla f = (f_x, f_y, f_z).$$

A vector field, $\mathbf{F} = (F_1, F_2, F_3)$, on $\mathbb{R}^3$, which is a map from an open subset of $\mathbb{R}^3$ to $\mathbb{R}^3$, whose coordinate functions are continuously differentiable has divergence

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = (F_1)_x + (F_2)_y + (F_3)_z.$$

Then, of course, $\Delta f = \operatorname{div} \operatorname{grad} f$ and we also use the notation

$$|\nabla f|^2 = f_r^2 + f_y^2 + f_z^2,$$

the norm of the gradient of f squared.

We'll diverge from the notation in the text slightly and write

$$\int_D f \, d\mathbf{z}$$

for the integral of the real-valued function f over the region $D \subseteq \mathbb{R}^3$ with the Euclidean volume element $d\mathbf{z}$. We'll write

$$\int_S h \, dS$$

to mean the integral of the real-valued function h, defined on the surface $S \subseteq \mathbb{R}^3$ with area element dS. Hopefully it'll be clear from context what we're integrating over.

A connection between integrating over volumes and surfaces is given by the **Divergence Theorem**. If D is bounded, open, whose boundary is, say, given by a level set of a continuously differentaible function of three variables whose gradient never vanishes, if $\mathbf{F}$ is a vector field with coordinate functions continuously differentiable, then

$$\int_D \operatorname{div} \mathbf{F} d\mathbf{z} = \int_{\text{bdy } D} \mathbf{F} \cdot \mathbf{n} dS.$$

with $\mathbf{n}$ the outward pointing unit normal of bdy D. If you wish, here's a proof for some fairly general regions D: http://www.math.ncku.edu.tw/ rchen/Advanced%20Calculus/divergence%20theorem.pdf

But the divergence theorem is also a particular case of Stokes' Theorem for differential forms. In any case, every integral identity for us is essentially a consequence of the divergence theorem.

5.1 Green's First Identity

Theorem 5.1. Green's First Identity

If v has continuous first partials and u has continuous second partials on D, then

$$\int_{\text{bdy } D} v \nabla u \cdot \mathbf{n} dS = \int_D \nabla v \cdot \nabla u d\mathbf{z} + \int_D v \Delta u d\mathbf{z}.$$

We have, for a function f and a vector field $\mathbf{F}$, $\operatorname{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$, so that

$$\operatorname{div}(v\nabla u) = \nabla v \cdot \nabla u + v \Delta u.$$

Then by the divergence theorem,

$$\int_{\text{bdy } D} v\nabla u \cdot \mathbf{n} dS = \int_D \text{div}(v\nabla u) d\mathbf{z}.$$

■

Theorem 5.2. Mean Value Property

The average value of a harmonic function over a sphere equals its value at the center.

A sphere, S, in $\mathbb{R}^3$ centered at the origin with radius a are solutions of the level set $|\mathbf{z}| = a$. If $D = \{\mathbf{z} \in \mathbb{R}^3 \mid |\mathbf{z}| < a\}$, then $S = \text{bdy } D$. Now, the outward unit normal of S is $\mathbf{n} = \frac{\mathbf{z}}{a}$ if $a = |\mathbf{z}|$. If you wish, $\mathbf{n}$ is the normalized gradient of the function $|\mathbf{z}|^2$. Then $\nabla u \cdot \mathbf{n} = u_r$ by the Chain Rule, with u_r the partial derivative of u with respect to the spherical coordinate r.

We use Green's First Identity with u harmonic and $v = 1$ to obtain

$$\int_S u_r dS = 0.$$

In spherical coordinates, this integral becomes

$$\int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin\phi d\phi d\theta = 0.$$

Hence,

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) \sin\phi d\phi d\theta = 0.$$

This is in fact true for any $r \leq a$, and we can again differentiate under the integral sign and conclude

$$\frac{1}{4\pi} \cdot \frac{\partial}{\partial} r \left(\int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\phi d\phi d\theta \right) = 0$$

for all $r \leq a$.

Hence,

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(a, \theta, \phi) \sin\phi d\phi d\theta = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\phi d\phi d\theta$$

for any $r \leq a$.

Now,

$$\lim_{r \to 0} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\phi d\phi d\theta = u(0)$$

from the continuity of u. Finally,

$$\frac{1}{\text{Area}(S)} \int_S u dS = u(0).$$

■

We now have

Theorem 5.3. The maximum principle

Any nonconstant harmonic function on a connected, open set only attains its extrema on the boundary.

The proof is the same as in the $\mathbb{R}^n$ case now that we have the mean value property.

5.2 Dirichlet's Principle

Dirichlet's Principle We define a certain energy functional on an open, bounded set D:

$$E(w) = \frac{1}{2}\int_D |\nabla w|^2 d\mathbf{z}$$

for all continuously differentiable functions w.

Theorem 5.4. If u is the unique harmonic function on D such that $u = h$ on bdy D, then for all w, continuously differentiable, with $w = h$ on bdy D, then

$$E(u) \leq E(w).$$

Fix such a w. Write $v = u - w$. Then $w = u - v$ and $|\nabla w|^2 = |\nabla u|^2 - 2\nabla u \cdot \nabla v + |\nabla v|^2$. So that

$$E(w) = E(u) - \int_D \nabla v \cdot \nabla u d\mathbf{z} + E(v).$$

Again we recall

$$\int_{\text{bdy } D} v\nabla u \cdot \mathbf{n} dS = \int_D \nabla v \cdot \nabla u d\mathbf{z} + \int_D v\Delta u d\mathbf{z}.$$

Since $v = 0$ on bdy D and u is harmonic on D, it follows

$$E(u) + E(v) = E(w).$$

Since $E(v) \geq 0$, it follows

$$E(u) \leq E(w).$$

5.3 Green's Second Identity

Theorem 5.5. Green's Second Identity

If u and v both have continuous second partials, then

$$\int_D (u\Delta v - v\Delta u) d\mathbf{z} = \int_{\text{bdy } D} (u\nabla v \cdot \mathbf{n} - v\nabla u \cdot \mathbf{n})\, dS.$$

We write Green's First Identity twice

$$\int_{\text{bdy } D} u\nabla v \cdot \mathbf{n} dS = \int_D \nabla u \cdot \nabla v d\mathbf{z} + \int_D u\Delta v d\mathbf{z},$$

$$\int_{\text{bdy } D} v\nabla u \cdot \mathbf{n} dS = \int_D \nabla v \cdot \nabla u d\mathbf{z} + \int_D v\Delta u d\mathbf{z},$$

and subtract.

Theorem 5.6. Representation Formula

If u is harmonic on $D \subseteq \mathbb{R}^3$, then

$$u(\mathbf{z}_0) = \frac{1}{4\pi}\int_{\text{bdy } D} -u\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{z}_0|}\right)\cdot\mathbf{n} + \frac{1}{|\mathbf{x}-\mathbf{z}_0|}\nabla u\cdot\mathbf{n}\,dS(\mathbf{x})$$

for all $\mathbf{z}_0 \in D$.

Fix $\varepsilon > 0$. And let $D_\varepsilon = D \setminus B_\varepsilon(\mathbf{z}_0)$.

wlog, let $\mathbf{z}_0$ be the origin, then $v = -1/(4\pi r) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{z}_0|}$ is a fundamental solution of Laplace's equation. In particular, v is harmonic on $\mathbb{R}^3\setminus\{0\}$. Then, with Green's second identity on D_ε with u and v, it follows

$$-\int_{\text{bdy } D_\varepsilon} (u\nabla(1/r)\cdot\mathbf{n} - (1/r)\nabla u\cdot\mathbf{n})\,dS = 0.$$

But bdy D_ε = bdy $D \cup$ bdy $B_\varepsilon(0)$. And the outward pointing unit normal of bdy D_ε along bdy $B_\varepsilon(0)$ is in fact $-\mathbf{z}/\varepsilon$, and again $\nabla u\cdot(-\mathbf{z}/\varepsilon) = -u_r$ by the Chain Rule. Hence,

$$\begin{aligned}
-\int_{\text{bdy } D_\varepsilon} &(u\nabla(1/r)\cdot\mathbf{n} - (1/r)\nabla u\cdot\mathbf{n})\,dS \\
&= -\int_{\text{bdy } D} (u\nabla(1/r)\cdot\mathbf{n} - (1/r)\nabla u\cdot\mathbf{n})\,dS \\
&\quad + \int_{\text{bdy } B_\varepsilon(0)} \left(u(1/r)_r - (1/r)u_r\right)dS.
\end{aligned}$$

It remains to show

$$\int_{\text{bdy } B_\varepsilon(0)} \left(u(1/r)_r - (1/r)u_r\right)dS \to -4\pi u(0)$$

as $\varepsilon \to 0$.

On $r = \varepsilon$, which is bdy $B_\varepsilon(0)$, $(1/r)_r = -(1/\varepsilon^2)$. Hence,

$$\begin{aligned}
-\int_{\text{bdy } B_\varepsilon(0)} &\left(u(1/r)_r - (1/r)u_r\right)dS \\
&= \frac{1}{\varepsilon^2}\int_{\text{bdy } B_\varepsilon(0)} u\,dS + \frac{1}{\varepsilon}\int_{\text{bdy } B_\varepsilon(0)} u_r\,dS \\
&= 4\pi\overline{u} + 4\pi\,\varepsilon\,\overline{u_r}.
\end{aligned}$$

Here, $\overline{u}$ is the average of u on bdy $B_\varepsilon(0)$. Similarly with $\overline{u_r}$. Again, since u is continuous, it follows

$$4\pi\overline{u} \to u(0)$$

as $\varepsilon \to 0$. Since u_r is bounded, it follows $4\pi\,\varepsilon\,\overline{u_r} \to 0$ as $\varepsilon \to 0$. The result follows.

The analogous result in $\mathbb{R}^n$ is obtained by replacing $\frac{1}{|\mathbf{x}-\mathbf{z}_0|}$ with $\log|\mathbf{x}-\mathbf{z}_0|$, the sign, and 4π with 2π. If $D \subseteq \mathbb{R}^n$, u is harmonic on D, then

$$u(\mathbf{z}_0) = \frac{1}{2\pi}\int_{\text{bdy } D} u\nabla\log|\mathbf{x}-\mathbf{z}_0|\cdot\mathbf{n} - \log|\mathbf{x}-\mathbf{z}_0|\nabla u\cdot\mathbf{n}\,dS(\mathbf{x}).$$

5.4 Green's Functions

Definition 5.7. The **Green's Function** for the operator $-\Delta$ and domain D at the point $\mathbf{z}_0 \in D$, is a function on $\overline{D}$ such that

1. $G(\mathbf{z}, \mathbf{z}_0)$ has continuous second partials with respect to $\mathbf{z}$ and $\Delta G = 0$ in D with respect to $\mathbf{z}$ except at $\mathbf{z}_0$.
2. $G(\mathbf{z}, \mathbf{z}_0) = 0$ when $\mathbf{z} \in \text{bdy } D$.
3. $G(\mathbf{z}, \mathbf{z}_0) + \frac{1}{4\pi|\mathbf{z}-\mathbf{z}_0|}$ is finite as $\mathbf{z} \to \mathbf{z}_0$, has continuous second partials in D, and is harmonic in D.

Theorem 5.8. If G is the Green's function for D, $-\Delta$ and $\mathbf{z}_0 \in D$, then

$$u(\mathbf{z}_0) = \int_{\text{bdy } D} u(\mathbf{z}) \nabla G(\mathbf{z}, \mathbf{z}_0) \cdot \mathbf{n} \, dS(\mathbf{z})$$

is the solution of Dirichlet's problem for harmonic functions on D.

Recall the representation formula

$$u(\mathbf{z}_0) = \int_{\text{bdy } D} (u \nabla v \cdot \mathbf{n} - v \nabla u \cdot \mathbf{n}) \, dS$$

with $v(\mathbf{z}) = -(4\pi|\mathbf{z} - \mathbf{z}_0|)^{-1}$. Define $H(\mathbf{z}) = G(\mathbf{z}, \mathbf{z}_0) - v(\mathbf{z})$. Then by item 3 above, H is harmonic in D. Green's second identity then states

$$0 = \int_{\text{bdy } D} (u \nabla H \cdot \mathbf{n} - H \nabla u \cdot \mathbf{n}) \, dS.$$

Since $H + v = G$, and $G = 0$ on bdy D, it follows

$$u(\mathbf{z}_0) = \int_{\text{bdy } D} u(\mathbf{z}) \nabla G(\mathbf{z}, \mathbf{z}_0) \cdot \mathbf{n} \, dS(\mathbf{z}).$$

■

The first two theorems show finding the Green's function for a domain is equivalent to solving the Dirichlet problem for harmonic functions on that domain. Soon we will make this a bit more precise.

Theorem 5.9. The existence of solutions of Dirichlet's problem for harmonic functions imply the existence of Green's functions.

Suppose $\mathbf{z}_0 \in D$, u is the unique solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } D \\ u(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{z}_0|} & \text{for all } \mathbf{x} \in \text{bdy } D \end{cases}$$

If we define

$$G(\mathbf{z}, \mathbf{z}_0) = u(\mathbf{z}) - \frac{1}{4\pi|\mathbf{z} - \mathbf{z}_0|}$$

for all $\mathbf{z} \neq \mathbf{z}_0 \in D$, then since $-\frac{1}{4\pi|\mathbf{z}-\mathbf{z}_0|}$ is a fundamental solution solution of the Laplacian, then G is certainly a Green's function of $-\Delta$ on D for $\mathbf{z}_0 \in D$.

■

Theorem 5.10. Symmetry of the Green's function

If D is any region with Green's function G,

$$G(\mathbf{z}, \mathbf{z}_0) = G(\mathbf{z}_0, \mathbf{z})$$

for all $\mathbf{z} \neq \mathbf{z}_0 \in D$.

Suppose $\mathbf{a}, \mathbf{b} \in D$ and are distinct. Fix $\varepsilon > 0$. This time, define

$$D_\varepsilon = D \setminus \left(\overline{B_\varepsilon(\mathbf{a})} \cup \overline{B_\varepsilon(\mathbf{b})} \right).$$

Then

$$\text{bdy } D_\varepsilon = \text{bdy } D \cup \text{bdy } B_\varepsilon(\mathbf{a}) \cup \text{bdy } B_\varepsilon(\mathbf{b}).$$

Then by the first item of the Green's function, if $u(\mathbf{z}) = G(\mathbf{z}, \mathbf{a})$ and $v(\mathbf{z}) = G(\mathbf{z}, \mathbf{b})$, then u and v are both harmonic in D_ε. By item 2, $u = v = 0$ on bdy D.

Green's second identity on D_ε is

$$\int_{D_\varepsilon} (u\Delta v - v\Delta u) d\mathbf{z} = \int_D (u\nabla v \cdot \mathbf{n} - v\nabla u \cdot \mathbf{n})\, dS + A_\varepsilon + B_\varepsilon$$

with

$$A_\varepsilon = \int_{\text{bdy } B_\varepsilon(\mathbf{a})} (u\nabla v \cdot \mathbf{n} - v\nabla u \cdot \mathbf{n})\, dS$$

and

$$B_\varepsilon = \int_{\text{bdy } B_\varepsilon(\mathbf{b})} (u\nabla v \cdot \mathbf{n} - v\nabla u \cdot \mathbf{n})\, dS.$$

By our considerations, the first two terms in Green's identity vanish. So,

$$A_\varepsilon + B_\varepsilon = 0.$$

Again define $H(\mathbf{z}) = G(\mathbf{z}, \mathbf{a}) + (4\pi|\mathbf{z} - \mathbf{a}|)^{-1}$ and $r = |\mathbf{z} - \mathbf{a}|$. Then,

$$u = H - \frac{1}{4\pi r}.$$

So,

$$A_\varepsilon = \int_{\text{bdy } B_\varepsilon(\mathbf{a})} \left(\left(H - \frac{1}{4\pi r} \right) \nabla v \cdot \mathbf{n} - v\nabla \left(H - \frac{1}{4\pi r} \right) \cdot \mathbf{n} \right) dS.$$

In spherical coordinates, this integral is

$$\int_0^{2\pi} \int_0^{\pi} \left(-\left(H - \frac{1}{4\pi\, \varepsilon} \right) v_r + v \left(H - \frac{1}{4\pi r} \right)_r \right) \varepsilon^2 \sin\phi d\phi d\theta.$$

Again, on bdy $B_\varepsilon(\mathbf{a})$, $r = \varepsilon$ and $\nabla f \cdot \mathbf{n} = -f_r$ for any function f on bdy D_ε along bdy $B_\varepsilon(\mathbf{a})$.

Since H, H_r are continuous from item 3, and v, v_r are continuous near $\mathbf{a}$ by item 1, it follows, as $\varepsilon \to 0$, every term in A_ε but the last vanishes because of the ε^2 factor from the Jacobian. Again, $(1/r)_r = -1/\varepsilon^2$ if $r = \varepsilon$. So,

$$\lim_{\varepsilon \to 0} A_\varepsilon = \lim_{\varepsilon \to 0} \frac{1}{4\pi\, \varepsilon^2} \int_{\text{bdy } B_\varepsilon(\mathbf{a})} v dS = v(\mathbf{a}).$$

Again define $H(\mathbf{z}) = G(\mathbf{z}, \mathbf{b}) + (4\pi|\mathbf{z} - \mathbf{b}|)^{-1}$ and $r = |\mathbf{z} - \mathbf{b}|$. Then,

$$v = H - \frac{1}{4\pi r}.$$

So,

$$B_\varepsilon = \int_{\text{bdy } B_\varepsilon(\mathbf{b})} \left(u\nabla \left(H - \frac{1}{4\pi r} \right) \cdot \mathbf{n} - \left(H - \frac{1}{4\pi r} \right) \nabla u \cdot \mathbf{n} \right) dS.$$

In spherical coordinates, this integral is

$$\int_0^{2\pi} \int_0^{\pi} \left(-u \left(H - \frac{1}{4\pi r} \right)_r + \left(H - \frac{1}{4\pi \varepsilon} \right) u_r \right) \varepsilon^2 \sin\phi d\phi d\theta.$$

Again, on bdy $B_\varepsilon(\mathbf{b})$, $r = \varepsilon$ and $\nabla f \cdot \mathbf{n} = -f_r$ for any function f on bdy D_ε along bdy $B_\varepsilon(\mathbf{b})$.

Since H, H_r are continuous from item 3, and u, u_r are continuous near $\mathbf{b}$ by item 1, it follows, as $\varepsilon \to 0$, every term in B_ε but the second vanishes because of the ε^2 factor from the Jacobian. Again, $(1/r)_r = -1/\varepsilon^2$ if $r = \varepsilon$. So,

$$\lim_{\varepsilon \to 0} B_\varepsilon = -\lim_{\varepsilon \to 0} \frac{1}{4\pi\,\varepsilon^2} \int_{\text{bdy } B_\varepsilon(\mathbf{b})} u dS = -u(\mathbf{b}).$$

Finally, since $A_\varepsilon + B_\varepsilon = 0$ and since $\varepsilon > 0$ is arbitrary, it follows

$$v(\mathbf{a}) - u(\mathbf{b}) = 0.$$

That is,

$$G(\mathbf{a}, \mathbf{b}) = G(\mathbf{b}, \mathbf{a}).$$

Since $\mathbf{a} \neq \mathbf{b} \in D$ are arbitrary, the theorem follows.

■

This theorem means we can think of Green's function as a continuous function on $\overline{D} \times \overline{D} \setminus \{(x, x) \mid x \in \overline{D}\}$, with $G(\mathbf{z}_0, \mathbf{z}) = G(\mathbf{z}, \mathbf{z}_0)$ if $\mathbf{z}_0 \in \overline{D}, \mathbf{z} \in \overline{D}$, $\mathbf{z}_0 \neq \mathbf{z}$.

Theorem 5.11. If h is continuous on bdy D and if G is the Green's function for D, $-\Delta$ and $\mathbf{z}_0 \in D$, and is smooth on $\overline{D} \times \overline{D} \setminus \{(x, x) \mid x \in \overline{D}\}$, then

$$u(\mathbf{z}_0) = \int_{\text{bdy } D} h(\mathbf{x}) \nabla G(\mathbf{x}, \mathbf{z}_0) \cdot \mathbf{n} dS(\mathbf{x})$$

has the properties $\Delta u = 0$ in D and $u = h$ on bdy D.

Fix $\mathbf{x} \in$ bdy D and $\mathbf{z}_0 \in D$. Then $\nabla G(\mathbf{x}, \mathbf{z}_0)$ is continuous at $(\mathbf{x}, \mathbf{z}_0)$ and harmonic with respect to $\mathbf{z}_0$, since $G(\mathbf{x}, \mathbf{z}_0) = G(\mathbf{z}_0, \mathbf{x})$. Hence, since h is also continuous, we differentiate under the integral sign and conclude u is harmonic on D.

It takes a bit more work to prove $\lim_{\mathbf{z}_0 \to \mathbf{x}} u(\mathbf{z}_0) = h(\mathbf{x})$. With a similar amount of effort, one can also prove that a Green's function exists for a given domain and is smooth like in the above hypothesis. Therefore, showing the existence of the Green's functions is equivalent to solving the Dirichlet problem for harmonic functions.

5.5 Green's function on the upper half-space

Green's function on the upper half-space

The function $-\frac{1}{4\pi|\mathbf{z}-\mathbf{z}_0|}$ actually satisfies items 1 and 3 of the Green's function's properties. We just need to modify it so that $G = 0$ on bdy D. Consider $D = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, the upper half-space. Then D is open and bdy $D = \{z = 0\}$. What we do is move the singularity of the fundamental solution away from the domain while still trying to impose item 2. If $\mathbf{z} = (x, y, z)$, define $\overline{\mathbf{z}} = (x, y, -z)$, akin to complex conjugation, but is just reflection about bdy D. In particular, $\mathbf{z}$ and $\overline{\mathbf{z}}$ are the same distance from bdy D, namely, $|z|$. This implies, if $\mathbf{z} \in$ bdy D, then

$$|\mathbf{z}-\mathbf{z}_0| = |\mathbf{z}-\overline{\mathbf{z}_0}|.$$

Recall $|\mathbf{z}-\mathbf{z}_0|^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$.

Notice if $\mathbf{z} \in D$, then $\overline{\mathbf{z}} \notin D$. Hence, by translation invariance, the function $\frac{1}{4\pi|\mathbf{z}-\overline{\mathbf{z}_0}|}$ is harmonic on D as long as $\mathbf{z}_0 \in D$ as well. Moreover, the linearity of the Laplacian implies

$$G(\mathbf{z}, \mathbf{z}_0) = \frac{1}{4\pi|\mathbf{z}-\overline{\mathbf{z}_0}|} - \frac{1}{4\pi|\mathbf{z}-\mathbf{z}_0|}$$

satisfies items 1 and 3 of the Green's function. If $\mathbf{z} \in$ bdy D, then

$$G(\mathbf{z}, \mathbf{z}_0) = 0.$$

That is, G is the Green's function on the upper half-space D.

Now notice on bdy D, $\mathbf{n} = -e_3$. Then

$$\nabla G \cdot \mathbf{n} = -G_z = -\left(-\frac{z+z_0}{4\pi|\mathbf{z}-\overline{\mathbf{z}_0}|^3} + \frac{z-z_0}{4\pi|\mathbf{z}-\mathbf{z}_0|^3}\right) = \frac{z_0}{2\pi|\mathbf{z}-\mathbf{z}_0|^3}$$

on bdy D. In general, recall

$$u(\mathbf{z}_0) = \int_{\text{bdy } D} u(\mathbf{z}) \nabla G(\mathbf{z}, \mathbf{z}_0) \cdot \mathbf{n} \, dS(\mathbf{z})$$

for u the solution of Laplace's Dirichlet problem on D. Therefore, if $\Delta u = 0$ on $D = \{z > 0\}$ and $u = h$ on bdy $D = \{z = 0\}$, then

$$u(\mathbf{z}_0) = \frac{z_0}{2\pi} \int_{\{z=0\}} \frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{z}_0|^3} dS(\mathbf{x}),$$

a type of Poisson's formula for harmonic functions on the upper half-space.

Indeed, the formula

$$u(\mathbf{z}_0) = \int_{\text{bdy } D} u(\mathbf{z}) \nabla G(\mathbf{z}, \mathbf{z}_0) \cdot \mathbf{n} \, dS(\mathbf{z})$$

is essentially Possion's formula for an arbitrary domain, and in $\mathbb{R}^n$ one can show this reduces to Possion's formula for harmonic functions on a disk. On a sphere in $\mathbb{R}^3$, the formula takes a similar form, and is left to a reference. Say, [Strauss, '92 pg 193-195].

6 Distributions

Distributions We now introduce a notion of weak solution.

From [Strauss, '92 ch 12]

Definition 6.1.

- A **test function** is a real-valued function, defined on $\mathbb{R}^n$, C^∞ (smooth, all derivatives exist and are continuous), and which vanishes outside of some bounded set. Denote the space of all such functions $\mathcal{D}$. $\mathcal{D}$ is a vector space over $\mathbb{R}$.
- A **distribution**, f is a real-valued map $\phi \mapsto (f, \phi)$ defined on $\mathcal{D}$ with the following properties.
  1. **Linearity** $(f, a\phi + b\psi) = a(f, \phi) + b(f, \psi)$ for all $a, b \in \mathbb{R}$ and $\phi, \psi \in \mathcal{D}$.
  2. **continuity** If ϕ_n is a sequence of maps in $\mathcal{D}$, vanishing outside of a common interval, and this sequence and all of their derivatives converge uniformly to ϕ and its derivatives, respectively, then $(f, \phi_n) \to (f, \phi)$ as $n \to \infty$.

  We denote the space of all distributions by $\mathcal{D}'$ and note that $\mathcal{D}'$ is also a vector space, a kind of dual space of $\mathcal{D}$.

Example 6.2.

- We define the **Dirac delta function**, δ, as the distribution

$$(\delta, \phi) = \phi(0).$$

- If f is a **locally integrable** function on $\mathbb{R}^n$, meaning $\int_K f < \infty$ for any K closed and bounded, then define

$$(f, \phi) = \int_{\mathbb{R}^n} f\phi.$$

  f itself is a distribution.

If ϕ is any test function, we define, for any $x \in \mathbb{R}^n$,

$$\tau_x\phi(y) = \phi(x + y) \quad \text{and} \quad \tilde{\phi}(x) = \phi(-x).$$

If f is any distribution, we define the translation by x and reflection about the origin of f by

$$(\tau_x f, \phi) = (f, \tau_{-x}\phi) \quad \text{and} \quad (\widetilde{f}, \phi) = (f, \widetilde{\phi}).$$

We define the **convolution** of f with ϕ by

$$f * \phi(x) = (f, \widetilde{\tau_x\phi})$$

for all $x \in \mathbb{R}^n$. It can be shown $f * \phi$ is a smooth function on $\mathbb{R}^n$.

For example, if f is locally integrable on $\mathbb{R}^n$, then

$$f * \phi(x) = \int_{\mathbb{R}^n} f(y)\widetilde{\tau_x\phi}(y)dy = \int_{\mathbb{R}^n} f(y)\phi(x - y)dy.$$

There's a useful topology on $\mathcal{D}'$.

Definition 6.3. We say a sequence of distributions f_n **converges weakly** to $f \in \mathcal{D}'$ as $n \to \infty$ if, for any test function $\phi \in \mathcal{D}$,

$$(f_n, \phi) \to (f, \phi)$$

as $n \to \infty$.

For example, recall the Nth term of the Fourier series of ϕ on $[-\pi, \pi]$.

$$S_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} \phi(y) e^{-iny} dy \cdot e^{nix}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(y) \sum_{n=-N}^{N} e^{in(x-y)} dy.$$

If ϕ is a test function which vanishes outside of $(-\pi, \pi)$ and $K_N(x) = \sum_{n=-N}^{N} e^{inx}$ (called the Dirichlet kernel), then the pointwise convergence theorem of Fourier series implies

$$\frac{1}{2\pi} \int_{\mathbb{R}} \phi(y) K_N(x-y) dy \to \phi(x)$$

as $N \to \infty$.

We note

$$\frac{1}{2\pi} \int_{\mathbb{R}} \phi(x-y) K_N(y) dy \to \phi(x)$$

as $N \to \infty$ for all $x \in (-\pi, \pi)$. Then this states

$$K_N * \phi(x) \to 2\pi\phi(x)$$

as $N \to \infty$. Rather,

$$(\tau_{-x}\widetilde{K_N}, \phi) \to 2\pi\phi(x)$$

as $N \to \infty$. That is,

$$\tau_{-x}\widetilde{K_N} \to 2\pi\tau_{-x}\delta$$

weakly as $N \to \infty$. In particular, $K_N \to 2\pi\delta$ weakly as $N \to \infty$.

We define the derivative, f', of a distribution, f, as

$$(f', \phi) = -(f, \phi')$$

for any test function ϕ, akin to integration by parts. We define partial differentiation in the same way.

For example,

$$(\Delta f, \phi) = (f, \Delta\phi)$$

for any distribution f and any test function ϕ. Notice if f is twice differentiable, this is 'Green's second identity' on $\mathbb{R}^n$. The boundary terms vanish because ϕ is a test function and in particular vanishes outside of some bounded set.

From your homework, you showed

$$-\int_{\mathbb{R}^3} \frac{1}{4\pi r} \cdot \Delta\phi(\mathbf{x}) d\mathbf{x} = \phi(0)$$

if ϕ is a test function on $\mathbb{R}^3$.

This means

$$\Delta\left(-\frac{1}{4\pi r}\right) = \delta$$

as distributions on $\mathbb{R}^3$.

You also proved in your homework that the solution of

$$\Delta u = f \text{ in } D \quad \text{ and } \quad u = 0 \text{ on bdy } D$$

is

$$u(\mathbf{x}_0) = \int_D G_{\mathbf{x}_0}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$$

if $G_{\mathbf{x}_0}(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function on D. Now

$$(\tau_{-\mathbf{x}_0}\delta, u) = u(\mathbf{x}_0).$$

Also,

$$\int_D G_{\mathbf{x}_0}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = (\Delta G_{\mathbf{x}_0}, u)$$

if u is also a test function which vanishes outside of D.

Hence,

$$\Delta G_{\mathbf{x}_0} = \tau_{-\mathbf{x}_0}\delta$$

for test functions which vanish outside of D. Also, $G_{\mathbf{x}_0}$ is a function and

$$G_{\mathbf{x}_0} = 0$$

on bdy D.

Green's function is the unique distribution which solves Poisson's equation with a point source in D and with homogeneous Dirichlet boundary conditions.

We can now see why finding Green's function is equivalent to solving the Dirichlet problem for classical harmonic maps. But now we've shifted the work to showing these weak solutions are in fact smooth functions. However, our fundamental solution is explicit.

$$\Delta\left(-\frac{1}{4\pi r}\right) = \delta.$$

In your homework you show $\delta * f = f$ for test functions on $\mathbb{R}^3$. So that if we define $N = -\frac{1}{4\pi r}$ as a distribution and

$$u = N * f,$$

then

$$\Delta u = \Delta N * f = \delta * f = f.$$

That is, u solves Possion's problem with source f. Hence, the main issue is what happens on the boundary. As a note, one can show $(f * \phi)' = f' * \phi = f * \phi'$.

Last time, we showed the Green's function G on a domain D is the unique distribution solving the following Poisson's homogeneous Dirichlet boundary value problem.

$$\begin{cases} \Delta G_{\mathbf{x}_0} = \tau_{-\mathbf{x}_0}\delta & \text{in } D \\ G = 0 & \text{on bdy } D. \end{cases}$$

We can also ask similar questions about the wave and heat equations. For example, let's try to solve the problem

$$S_t = kS_{xx} \text{ in } \mathbb{R}\times(0,\infty) \quad \text{and} \quad S = \delta \text{ on } \mathbb{R}\times\{0\}.$$

First, we need a lemma.

From [Strauss, '92 3.5].

Lemma 6.4. If $S^t(x) = \frac{1}{\sqrt{4\pi kt}} \cdot e^{-r^2/4kt}$, then

$$\tau_{-x}\widetilde{S^t} \to \tau_{-x}\delta$$

weakly as $t \to 0^+$.

Remember, since $\tau_{-x}\widetilde{S^t}(y) = \widetilde{S^t}(y-x) = S^t(x-y)$, we must show

$$\int_{\mathbb{R}} \phi(x-y)S^t(y)dy = \int_{\mathbb{R}} \phi(y)\tau_{-x}\widetilde{S^t}(y)dy \to \phi(x)$$

as $t \to 0^+$ for every test function $\phi \in \mathcal{D}$.

Notice $\tau_{-x}\widetilde{S^t}(0) = S^t(x)$. So, if we define

$$S(x,t) = \tau_{-x}\widetilde{S^t}(0),$$

then S solves the heat equation on $\mathbb{R}\times(0,\infty)$. (verify this) Moreover, since $\widetilde{S^t} = S^t$, granting this lemma,

$$\int_{\mathbb{R}} \phi(y)S(y,t)dy \to \phi(0)$$

as $t \to 0^+$ follows. That is,

$$S \to \delta$$

weakly as $t \to 0^+$.

Let's prove the lemma. This is from [Strauss, '92 pg 79-80].

Remember, we must show

$$\int_{\mathbb{R}} \phi(x-y)S^t(y)dy \to \phi(x)$$

as $t \to 0^+$ for every test function $\phi \in \mathcal{D}$.

Recall

$$S^t(y) = \frac{1}{\sqrt{4\pi kt}} \cdot e^{-y^2/4kt}.$$

Let $p = y/\sqrt{kt}$. Then

$$\int_{\mathbb{R}} \phi(x-y) S^t(y) dy = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt}) dp.$$

Write

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt}) dp.$$

You should prove (this is an exercise in Strauss and Haberman) that $\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} dp = 1$. One way to show this is to consider the integral $\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(r^2+y^2)/4} dx dy$ in polar coordinates and use Fubini's theorem.

Given this,

$$u(x,t) - \phi(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \left(\phi \left(x - p\sqrt{kt} \right) - \phi(x) \right) dp$$

Fix $x \in \mathbb{R}$. Fix $\varepsilon > 0$. The continuity of ϕ implies there is some $\rho > 0$ such that

$$\max_{|x-y| \le \rho} |\phi(y) - \phi(x)| < \frac{\varepsilon}{2}.$$

Notice

$$\begin{aligned} &\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \left(\phi \left(x - p\sqrt{kt} \right) - \phi(x) \right) dp \\ &\quad = \frac{1}{\sqrt{4\pi}} \int_{|p| \le \rho/\sqrt{kt}} e^{-p^2/4} \left(\phi \left(x - p\sqrt{kt} \right) - \phi(x) \right) dp \\ &\quad\quad + \frac{1}{\sqrt{4\pi}} \int_{|p| > \rho/\sqrt{kt}} e^{-p^2/4} \left(\phi \left(x - p\sqrt{kt} \right) - \phi(x) \right) dp \end{aligned}$$

If $w(p,x,t) = e^{-p^2/4} \left(\phi \left(x - p\sqrt{kt} \right) - \phi(x) \right)$,

$$\begin{aligned} \left| \frac{1}{\sqrt{4\pi}} \int_{|p| \le \rho/\sqrt{kt}} w(p,x,t) dp \right| &\le \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} e^{-p^2/4} dp \max_{|x-y| \le \rho} |\phi(y) - \phi(x)| \\ &< 1 \cdot \frac{\varepsilon}{2}. \end{aligned}$$

And

$$\left| \frac{1}{\sqrt{4\pi}} \int_{|p| > \rho/\sqrt{kt}} w(p,x,t) dp \right| \le \frac{1}{\sqrt{4\pi}} \cdot 2 \max_{\mathbb{R}} |\phi| \int_{|p| > \rho/\sqrt{kt}} e^{-p^2/4} dp.$$

Now, since ϕ is a test function, it's bounded and so its maximum indeed exists. If we make t sufficiently small, then we've made $\rho/\sqrt{kt}$ sufficiently large. We show, if $\int_{\mathbb{R}} f(x) dx < \infty$, then

$$\lim_{N \to \infty} \int_{|x| > N} f(x) dx = 0,$$

completing the proof.

This is the continuous analogue of saying the tail end of a series approaches zero.

Let's assume

$$\int_{\mathbb{R}} f(x)dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)dx,$$

which is true for $f(x) = e^{-r^2/4}$.

Define $a_N = \int_{-N}^{N} f(x)dx$. Then

$$\left| \int_{|x|>N} f(x)dx \right| = \left| \int_{\mathbb{R}} f(x)dx - a_N \right| \to 0$$

as $N \to \infty$.

To complete the proof of the lemma, we notice

$$\frac{1}{\sqrt{4\pi}} \cdot 2 \max_{\mathbb{R}} |\phi| \int_{|p|>\rho/\sqrt{kt}} e^{-p^2/4} dp < \frac{\varepsilon}{2}$$

if t is sufficiently small.

That is,

$$|u(x,t) - \phi(x)| < \varepsilon$$

if t is sufficiently small.

Since ϕ is an arbitrary test function, we've shown

$$\int_{\mathbb{R}} \phi(x-y) S^t(y) dy \to \phi(x)$$

as $t \to 0^+$ for every test function $\phi \in \mathcal{D}$. That is,

$$\tau_{-x} \widetilde{S^t} \to \tau_{-x} \delta$$

weakly as $t \to 0^+$.

Last time, we showed the solution of

$$S_t = kS_{xx} \text{ in } \mathbb{R} \times (0,\infty) \quad \text{and} \quad S = \delta \text{ on } \mathbb{R} \times \{0\}$$

is

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \cdot e^{-r^2/4kt}.$$

In particular, we showed $S \to \delta$ weakly as $t \to 0^+$.

We can then find a solution of

$$u_t = ku_{xx} \text{ in } \mathbb{R} \times (0,\infty) \quad \text{and} \quad u = \phi \text{ on } \mathbb{R} \times \{0\}$$

if ϕ is a test function by defining

$$u = S * \phi.$$

Then

$$u_t = S_t * \phi = kS_{xx} * \phi = ku_{xx}.$$

And $u \to \delta * \phi = \phi$ weakly as $t \to 0^+$. But then weak convergence of locally integrable functions to a locally integrable function implies pointwise convergence of the sequence to the function. Hence,

$$u = \phi$$

on $\mathbb{R} \times \{0\}$. If $t > 0$,

$$u(x,t) = \int_{\mathbb{R}} S(x-y)\phi(y)dy.$$

This is exactly the result obtained in [Strauss, '92 ch 2-3].

An **absolutely integrable function** f on $\mathbb{R}$ is one for which $\int_{\mathbb{R}} |f| < \infty$. Let's turn our attention to the **Fourier transform** of an absolutely integrable function f:

$$\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-2\pi ixk}dx.$$

It turns out, if f is continuous and absolutely integrable, then

$$f(x) = \int_{\mathbb{R}} \hat{f}(k)e^{2\pi ikx}dk.$$

The next few slides are from [Folland, '95 ch 0D].

First,

$$\int_{\mathbb{R}} e^{-\pi r^2}dx = 1 \tag{6.5}$$

as proved again by using polar coordinates and Fubini's theorem for the integrand $e^{-\pi(r^2+y^2)}$.

Next, we prove the Fourier transform of a Gaussian is another Gaussian. That is, if $f(x) = e^{-\pi r^2}$, then

$$\hat{f}(k) = e^{-\pi k^2}.$$

We write

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-\pi r^2}e^{-2\pi ixk}dx = e^{-\pi k^2}\int_{\mathbb{R}} e^{-\pi(x+ik)^2}dx.$$

Then we change variables by shifting the contour $y = x + ik$ and use equation (6.5) to conclude the result.

Now, for any positive a, if $f_a(x) = e^{-\pi ar^2}$, then

$$\hat{f}_a(k) = a^{-1/2}e^{-\pi k^2/a} \tag{6.6}$$

Recall we have

$$\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-2\pi ixk}dx.$$

If $y = a^{-1/2}x$, then $x = a^{1/2}y$ and $dx = a^{1/2}dy$. Then if $f(x) = e^{-\pi r^2}$, then

$$e^{-\pi k^2} = \int_{\mathbb{R}} f(x)e^{-2\pi ixk}dx = a^{1/2}\int_{\mathbb{R}} f_a(y)e^{-2\pi iya^{1/2}k}dy = a^{1/2}\hat{f}_a(a^{1/2}k).$$

Equation (6.6) follows.

If f, g and fg are all continuous and absolutely integrable, then

$$\int_{\mathbb{R}} f\widehat{g} = \int_{\mathbb{R}} \widehat{f} g. \tag{6.7}$$

By Fubini's theorem,

$$\int_{\mathbb{R}} f\widehat{g} = \int_{\mathbb{R}}\int_{\mathbb{R}} f(k)g(x)e^{-2\pi ixk}dxdk = \int_{\mathbb{R}}\int_{\mathbb{R}} f(k)e^{-2\pi ixk}dkg(x)dx$$

$$= \int_{\mathbb{R}} \widehat{f} g.$$

We also need the lemma from lecture 19. If f is continuous and absolutely integrable and

$$g_\varepsilon(x) = \varepsilon^{-1}\, e^{-\pi\,\varepsilon^{-2}\,r^2},$$

then

$$g_\varepsilon * f \to f \text{ as } \varepsilon \to 0 \text{ pointwise on } \mathbb{R}\,. \tag{6.8}$$

We can now prove the

Theorem 6.9. Fourier Inversion Theorem
If f is continuous and absolutely integrable, then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(k)e^{2\pi ikx}dk$$

for all $x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Define

$$\phi(k) = e^{2\pi ixk - \pi\,\varepsilon^2\,k^2}.$$

Then

$$\widehat{\phi}(y) = \int_{\mathbb{R}} e^{-2\pi i(y-x)k}e^{-\pi\,\varepsilon^2\,k^2}dk = \varepsilon^{-1}\,e^{-\pi(x-y)^2/\,\varepsilon^2} = g_\varepsilon(x-y)$$

by (6.6).

Notice ϕf is absolutely integrable if f is, since $|\phi f| \le |f|$. Hence, by (6.7) and (6.8),

$$\int \phi\widehat{f} = \int \widehat{\phi} f = g_\varepsilon * f \to f \text{ as } \varepsilon \to 0.$$

But also,

$$\int \phi\widehat{f} = \int e^{2\pi ixk - \pi\,\varepsilon^2\,k^2}\widehat{f}(k)dk \to \int_{\mathbb{R}} \widehat{f}(k)e^{2\pi ikx}dk \text{ as } \varepsilon \to 0.$$

That is,

$$f(x) = \int_{\mathbb{R}} \widehat{f}(k)e^{2\pi ikx}dk$$

for all $x \in \mathbb{R}$.

Last time, we defined, for an absolutely integrable function f, the **Fourier transform** of f

$$\widehat{f}(k) = \int_{\mathbb{R}} f(x)e^{-2\pi ixk}dx.$$

We showed, if f is continuous and absolutely integrable, then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(k) e^{2\pi i k x} dk.$$

In doing so, we showed

$$\widehat{f}(k) = a^{-1/2} e^{-\pi k^2/a}$$

for all $k \in \mathbb{R}$ if $f(x) = e^{-\pi a r^2}$ for all $x \in \mathbb{R}$.

We list some properties of the Fourier transform which you should verify.

Function	Transform
f'	$2\pi i k \widehat{f}$
xf	$\frac{i}{2\pi}(\widehat{f})'$
$\tau_{-a} f$	$e^{-2\pi i a k} \widehat{f}$
$e^{2\pi i a x} f$	$\tau_{-a} \widehat{f}$
$af + bg$	$a\widehat{f} + b\widehat{g}$

We can also define the Fourier transform for some distributions. Not all, because the Fourier transform of a test function is not necessarily a test function. For example, $f(x) = \begin{cases} e^{-1/(1-r^2)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$ is a test function on $\mathbb{R}$ (verify this) but $\widehat{f}$ is not (as a complex-valued function). What people usually do is expand the space of test functions to the class of **Schwartz functions**. Then the dual space of the Schwartz functions is smaller than the space of distributions and is called the space of **tempered distributions**. For us, all that's important is that locally integrable functions and the Dirac delta function δ are tempered distributions, so they each have a well-defined Fourier transform.

We define

$$(\widehat{\delta}, \phi) = (\delta, \widehat{\phi}) = \widehat{\phi}(0)$$

for each Schwartz function ϕ. Notice

$$\widehat{\phi}(0) = \int_{\mathbb{R}} \phi(x) dx = (1, \phi).$$

That is, the Fourier transform of δ is the constant function 1. Similarly,

$$(\widehat{1}, \phi) = (1, \widehat{\phi}) = \int_{\mathbb{R}} \widehat{\phi}(k) dk = \int_{\mathbb{R}} \widehat{\phi}(k) e^{2\pi i k \cdot 0} dk = \phi(0) = (\delta, \phi).$$

That is, the Fourier transform of 1 is the Dirac delta function δ.

There's another important distribution, called the Heaviside (or step) function.

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

H is locally integrable. We remark,

$$(H', \phi) = -(H, \phi') = -\int_{\mathbb{R}} H\phi' = -\int_0^\infty \phi' = -\phi|_0^\infty = \phi(0) = (\delta, \phi)$$

if ϕ is any test function. That is, $H' = \delta$ as distributions.

What's the Fourier transform of H? We define, for $a > 0$,

$$f_a(x) = \begin{cases} \frac{1}{2}e^{-2\pi ax} & \text{if } x > 0 \\ -\frac{1}{2}e^{2\pi ax} & \text{if } x < 0 \end{cases}$$

and note that $f_a \to H - \frac{1}{2}$ weakly as $a \to 0^+$. Moreover, f_a is absolutely integrable for all $a > 0$ and

$$\hat{f}_a(k) = \frac{1}{4\pi(a+ik)} + \frac{1}{4\pi(ik-a)}.$$

Then

$$(\hat{f}_a, \phi) = \int_{\mathbb{R}} + \frac{\hat{\phi}(k)}{4\pi(a+ik)} + \frac{\hat{\phi}(k)}{4\pi(ik-a)} dk \to \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{\phi}(k)}{ik} dk$$

as $a \to 0^+$. But also

$$(\hat{f}_a, \phi) = (f_a, \hat{\phi}) \to (H - \frac{1}{2}, \hat{\phi}) = (\widehat{H - \frac{1}{2}}, \phi)$$

as $a \to 0^+$.

Hence,

$$\hat{H}(k) = \frac{1}{2}\left(\delta - \frac{i}{\pi k}\right).$$

Let's find the solution of

$$\begin{cases} S_t = S_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ S = \delta & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We evaluate each side of the heat equation and the initial conditions under the Fourier transform in x. By the first property listed in the last lecture,

$$\hat{S}_t(k, t) = -(2\pi k)^2 \hat{S}(k, t)$$

on $\mathbb{R} \times (0, \infty)$. Furthermore,

$$\hat{S}(k, 0) = 1$$

on $\mathbb{R}$. For each $k \in \mathbb{R}$, this is an ODE.

Then

$$\hat{S}(k, t) = e^{-(2\pi k)^2 t}.$$

This is a Gaussian, whose inverse Fourier transform, which we've shown, is also a Gaussian.

$$\hat{f}(k) = a^{-1/2} e^{-\pi k^2/a}$$

for all $k \in \mathbb{R}$ if $f(x) = e^{-\pi a r^2}$ for all $x \in \mathbb{R}$. If $a = \frac{1}{4\pi t}$ and

$$S(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t},$$

then

$$\hat{S}(k, t) = e^{-(2\pi k)^2 t}.$$

S is exactly the source function we introduced earlier. We've shown $S \to \delta$ weakly as $t \to 0^+$. S is smooth on $\mathbb{R} \times (0, \infty)$ and certainly is a solution of $S_t = S_{xx}$.

Let's consider Laplace's equation on the upper-half plane with an impulse on the horizontal axis. We'll solve

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{on} \quad \mathbb{R} \times (0, \infty) \\ u &= \delta && \text{on} \quad \mathbb{R} \times \{0\}. \end{aligned}$$

We'll transform the x variable, to obtain

$$-(2\pi k)^2 \widehat{u} + \widehat{u}_{yy} = 0$$

on $\mathbb{R} \times (0, \infty)$ and $\widehat{u} = 1$ on $\mathbb{R} \times \{0\}$. This ODE has one solution which is bounded on $\mathbb{R} \times (0, \infty)$.

$$\widehat{u}(k, y) = e^{-2\pi |k| y}.$$

Then

$$u(x, y) = \int_{\mathbb{R}} e^{-2\pi |k| y} e^{2\pi i k x} dk.$$

Then

$$\begin{aligned} u(x, y) &= \int_0^\infty e^{2\pi(ix-y)k} dk + \int_{-\infty}^0 e^{2\pi(ix+y)k} dk \\ &= \frac{1}{2\pi} \left(\frac{1}{y - ix} + \frac{1}{y + ix} \right) \\ &= \frac{y}{\pi(r^2 + y^2)}. \end{aligned}$$

The prove of the fact

$$\frac{y}{\pi(r^2 + y^2)} \to \delta$$

weakly as $y \to 0^+$ is left as an exercise.

Let's find the solution of

$$\begin{cases} S_{tt} = c^2 S_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ S = 0 & \text{on } \mathbb{R} \times \{0\} \\ S_t = \delta & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We evaluate each side of the wave equation and the initial conditions under the Fourier transform in x. By the first property listed in the last lecture,

$$\widehat{S}_{tt}(k, t) = -(2\pi c k)^2 \widehat{S}(k, t)$$

on $\mathbb{R} \times (0, \infty)$. Furthermore,

$$\widehat{S}(k, 0) = 0$$

and

$$\widehat{S}_t(k, 0) = 1$$

on $\mathbb{R}$. For each $k \in \mathbb{R}$, this is an ODE.

This has the unique solution

$$\widehat{S}(k,t) = \frac{1}{2\pi ck}\sin(2\pi ckt) = \frac{e^{2\pi ickt} - e^{-2\pi ickt}}{4\pi ick}.$$

In lecture 21, we showed the Fourier transform of

$$\operatorname{sgn}(x) = H(x) - H(-x) = 2\left(H(x) - \frac{1}{2}\right)$$

is

$$\widehat{\operatorname{sgn}}(k) = \frac{1}{\pi ik}.$$

Notice

$$\frac{e^{2\pi ickt} - e^{-2\pi ickt}}{4\pi ick} = \frac{1}{4c}e^{2\pi ickt}\widehat{\operatorname{sgn}}(k) - \frac{1}{4c}e^{-2\pi ickt}\widehat{\operatorname{sgn}}(k)$$

Even though sgn is not absolutely integrable, it's locally integrable, and hence has Fourier transform as a tempered distribution. Note also the table of properties of the Fourier transform also for the Fourier transform of a tempered distribution.

Hence,

$$\widehat{S}(k,t) = \left(\widehat{\frac{\tau_{ct}\operatorname{sgn} - \tau_{-ct}\operatorname{sgn}}{4c}}\right)(k)$$

So, we suspect

$$S(x,t) = \frac{\operatorname{sgn}(x+ct) - \operatorname{sgn}(x-ct)}{4c}$$

for all $(x,t) \in \mathbb{R}\times(0,\infty)$. Indeed, we rewrite

$$\frac{\operatorname{sgn}(x+ct) - \operatorname{sgn}(x-ct)}{4c} = \frac{H(x+ct) - H(x-ct)}{2c} = \frac{H(c^2t^2 - r^2)}{2c}$$

if $t > 0$. The last step follows from the fact $c^2t^2 > r^2$ if and only if $-ct < x < ct$ if $c, t > 0$.

We check

$$S(x,0) = 0.$$

And notice

$$f_a(x) = \begin{cases} e^{-\frac{a}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is smooth, locally integrable, and converges weakly to H as $a \to 0^+$.

$$\int_{\mathbb{R}} (f_a)_t(x+ct)\phi(x)dx = c\int_{\mathbb{R}} (f_a)_x(x+ct)\phi(x)dx = c((\tau_{ct}f_a)_x, \phi)$$

$$= -c(\tau_{ct}f_a, \phi') = -c(f_a, \tau_{-ct}\phi') \to -c(H, (\tau_{-ct}\phi)') = c(\tau_{ct}\delta, \phi)$$

as $a \to 0^+$. This shows $(\tau_{ct}f_a)_t \to c\tau_{ct}\delta$ weakly as $a \to 0^+$ (as distributions on $\mathbb{R}$). Similarly,

$$\int_{\mathbb{R}} (f_a)_t(x-ct)\phi(x)dx \to -c(\tau_{-ct}\delta, \phi)$$

as $a \to 0^+$. Hence,

$$S_t = \delta \quad \text{on } \mathbb{R}\times\{0\}.$$

Also,

$$\int_{\mathbb{R}} (f_a)_{tt}(x+ct)\phi(x)dx = c^2((\tau_{ct} f_a)_{xx}, \phi) = c^2(f_a, (\tau_{-ct}\phi)'').$$

So, similarly,

$$\int_{\mathbb{R}} (f_a)_{tt}(x+ct)\phi(x)dx \to c^2(\tau_{ct}\delta', \phi),$$

$$\int_{\mathbb{R}} (f_a)_{tt}(x-ct)\phi(x)dx \to c^2(\tau_{-ct}\delta', \phi),$$

$$\int_{\mathbb{R}} (f_a)_{xx}(x+ct)\phi(x)dx \to (\tau_{ct}\delta', \phi)$$

and

$$\int_{\mathbb{R}} (f_a)_{xx}(x-ct)\phi(x)dx \to (\tau_{-ct}\delta', \phi)$$

as $a \to 0^+$. Hence,

$$S_{tt} = c^2 S_{xx}.$$

We remark $(\tau_a\phi)' = \tau_a\phi'$ if a is real and $'$ denotes differentiation with respect to x, which is independent of a. Define $g(x) = x + a$. Then $\tau_a\phi = \phi \circ g$. So, $(\tau_a\phi)'(x) = \phi' \circ g(x) = \phi'(x+a)$ by the chain rule. Also, $\tau_a\phi'(x) = \phi'(x+a)$ by definition.

7 Laplace Transform

We introduce the Laplace transform, where for our purposes we transform the time variable rather than the spatial one. If f is locally integrable on $[0, \infty)$, then we define, for every $s \geq 0$,

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt.$$

If f is also differentiable and there is some $M, k > 0$ such that $|f(t)| \leq Me^{-kt}$ for all $t \geq 0$, then

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

if $s > k$, which can be proved with integration by parts.

Similarly, if f also is twice differentiable and there is some $M_1, k_1 > 0$ such that $|f'(t)| \leq M_1 e^{-k_1 t}$ for all $t \geq 0$, then

$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$

if $s > \max\{k, k_1\}$.

We need one more property of the Laplace transform.

$$\mathcal{L}\{\tau_{-b}H\tau_{-b}f\}(s) = e^{-bs}\mathcal{L}\{f\}(s) \tag{7.1}$$

with H the Heaviside step function.

We can now solve the wave equation on the half-line with inhomogeneous boundary data.

$$u_{tt} = c^2 u_{xx} \quad \text{on } (0,\infty) \times (0,\infty)$$

with

$$u(0,t) = f(t), \quad u(x,0) = u_t(x,0) = 0.$$

We also assume $u(x,t) \to 0$ as $x \to \infty$. Then

$$s^2 \mathcal{L}\{u\}(x,s) = c^2 \mathcal{L}\{u\}_{xx}(x,s)$$

from the initial conditions.

Then

$$\mathcal{L}\{u\}(0,s) = \mathcal{L}\{f\}(s)$$

implies

$$\mathcal{L}\{u\}(x,s) = \mathcal{L}\{a\}(s)e^{-sx/c} + \mathcal{L}\{f-a\}(s)e^{sx/c}$$

for some function a. Then from (7.1),

$$u(x,t) = H\left(t - \frac{x}{c}\right) a\left(t - \frac{x}{c}\right) + H\left(t - \frac{x}{c}\right)(f-a)\left(t - \frac{x}{c}\right).$$

The uniqueness of the wave equation on the half-line actually implies $a = f$, so that

$$u(x,t) = H\left(t - \frac{x}{c}\right) f\left(t - \frac{x}{c}\right).$$

Notice this function is not C^2 if the first and second derivatives of f do not vanish at 0. In general, u is at best a distributional solution of this equation. The wave equation is different than the heat equation in this sense. The singularities of the initial conditions follow the solution.

8 The Wave Equation

The Wave Equation We consider the wave equation

$$u_{tt} = c^2 u_{xx}$$

on $\mathbb{R}^n$. The lack of boundary conditions makes this situation easier to study. Later, we will study the wave equation in higher spatial dimensions.

The wave equation in one spatial dimension is nice because we can factor the operator.

$$\left(\partial_t - c\frac{\partial}{\partial r^1}\right)\left(\partial_t + c\frac{\partial}{\partial r^1}\right) u = u_{tt} - c^2 u_{xx} = 0.$$

If

$$\left(\partial_t + c\frac{\partial}{\partial r^1}\right) u = v,$$

then

$$\left(\partial_t - c\frac{\partial}{\partial r^1}\right) v = 0.$$

These are two first order linear pde. From the first few lectures, $\left(\partial_t - c\frac{\partial}{\partial r^1}\right) v = 0$ implies

$$v = h(x + ct)$$

for some C^1 function h.

Notice v is constant along the lines $x + ct = l_0$ for $l_0 \in \mathbb{R}$ in the xt-plane. These lines, as in the first few lectures are called the **characteristics** of the pde. Also, if x is a function of t, which physically corresponds to a moving observer, then $\frac{d}{dt}v(x(t), t) = v_t + \frac{dx}{dt}v_x$. If

$$\frac{dx}{dt} = -c$$

(if the observer's velocity is $-c$), then from the pde $v_t - cv_x = 0$,

$$\frac{d}{dt}v(x(t), t) = 0.$$

This means an observer moving at velocity $-c$ will not notice any motion of the wave given by $v(x+ct)$, which is referred to as a traveling wave. In this case, it has speed c. Waves of the form $v(x - ct)$ are also traveling waves and move at speed c, but in the direction of increasing x. Any wave of the form $v(x + ct) + v(x - ct)$ is called a standing wave. (picture)

It remains to solve

$$u_t + c\frac{\partial}{\partial r^1}(u) = h(x + ct).$$

If $s \in \mathbb{R}$, $A = (c, 1)$, and $Z_y(s, y) = A$, then $Z(s, y) = (cy + s, y)$ is the unique solution of this ode with $Z(s, 0) = (s, 0)$. And $(u \circ Z)_y(y, s) = h(2cy + s)$, so that

$$(u \circ Z)(s, y) - (u \circ Z)(s, 0) = \int_0^y h(2cq + s)dq = \frac{1}{2c}\int_s^{2cy+s} h(q)dq.$$

Then $x = cy + s$ and $y = t$, so that $(u \circ Z)(s, 0) = g(x - ct)$ is some function of one variable, and

$$u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(q)dq + g(x - ct).$$

From here, if $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$, then $u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(q)dq + g(x - ct)$ implies

$$\phi(x) = g(x)$$

and

$$\psi(x) = h(x) - cg'(x).$$

Hence,

$$u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(q)dq + \frac{1}{2}(\phi(x + ct) + \phi(x - ct)).$$

This is called **d'Alembert's formula**. If ψ is C^1 and ϕ is C^2, then u is a classical solution of $u_{tt} = c^2u_{xx}$ with $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$. Notice u is constant along the lines $x \pm ct = l$ for constant $l \in \mathbb{R}$ are called the **characteristic lines** of the above wave operator $\partial_{tt} - c^2\partial_{xx}$.

For example, if $\phi = 0$ and $\psi = \sin$, then

$$u(x, t) = \frac{1}{2c}(\cos(x - ct) - \cos(x + ct)) = \frac{1}{c}\sin(x)\sin(ct).$$

The last step is the sum-to-product rule. We can check this directly since 0 and sin are smooth. $u_{tt}(x,t) = -c\sin(x)\sin(ct)$ and $u_{xx} = -\frac{1}{c}\sin(x)\sin(ct)$. Also, $u_t(x,t) = \sin(x)\cos(ct)$, so that the initial conditions are verified.

If ϕ is continuous, and ψ is locally integrable, then this formula still holds and the resultant function is a weak solution of the wave equation. For example, if

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a \end{cases}$$

and

$$\psi = 0,$$

then

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct))$$

solves the wave equation but is not C^2. It's just as regular as ϕ is though: piecewise smooth. You can check with this ϕ that u is still a weak solution of the wave equation.

https://www.desmos.com/calculator/r1irfmzdce

Causality and Energy From **d'Alembert's formula**

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(q)dq + \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right),$$

we notice $u(x_0,t_0)$, if $t_0 > 0$, only depends on ϕ and ψ at the points (x,t) for which

$$|x - x_0| \leq c(t_0 - t).$$

We call this cone the **domain of dependence** of (x,t). If $(x_0,0)$ lies on the line $t = 0$, then the set of all points (x,t) for which $(x_0,0)$ is in the domain of dependence of (x,t) is called the **domain of influence** of $(x_0,0)$.

$(x_0,0)$ is in the domain of dependence of (x,t) if and only if

$$|x - x_0| \leq ct,$$

another cone. (x,t) is in the domain of influence of $(x_0,0)$ if and only if $(x_0,0)$ is in the domain of dependence of (x,t).

https://www.desmos.com/calculator/zm8lrnvup9

The domain of influence of the closed interval $|x - x_0| \leq R$ is the frustum $|x - x_0| \leq R + ct$. Therefore, if ϕ and ψ vanish when $|x| > R$, then u vanishes when $|x| > R + ct$.

To note this, if $|x_0| > R + ct_0$ and $|x| \leq R + ct$, then (x,t) is not in the domain of dependence of (x_0,t_0). Otherwise, $|x - x_0| \leq c(t_0 - t)$, which implies

$$|x_0| \leq -ct + |x| + ct_0 \leq R + ct_0,$$

a contradiction.

That is, the domain of dependence of a point (x_0,t_0) with $|x_0| > R + ct_0$ doesn't intersect the domain of influence of $[-R, R]$. This point and this interval are not causally connected.

If $\phi = \psi = 0$ on $[-R, R]$ and $|x| \leq R + ct$, then $u(x,t) = 0$.

We define the **total energy** of a solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

as the sum of its kinetic and potential energy.

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} u_t^2 + c^2 \frac{\partial}{\partial r^1}(u)^2 dx.$$

Suppose u vanishes outside a bounded set. That is, suppose ϕ and ψ vanish outside a bounded interval. Then

$$\frac{d}{dt}E(u) = \int_{\mathbb{R}} u_t u_{tt} + c^2 \frac{\partial}{\partial r^1}(u) u_{xt} dx = \int_{\mathbb{R}} u_{tt} u_t - c^2 u_{xx} u_t dx = 0.$$

Differentiation under the integral sign is permitted if u is a classical solution of the wave equation, and the second to last step is integration by parts keeping in mind the boundary terms vanish. That is, $E(u)$ is constant with respect to t, and so is constant on the xt-plane. This is the **conservation of energy**.

Now suppose E and B are both vector fields on $\mathbb{R}^4$, (x, y, z, t), satisfying the Maxwell equations of electromagnetism.

$$E_t = c\nabla \times B$$

$$B_t = -c\nabla \times E$$

$$\operatorname{div}(E) = 0$$

$$\operatorname{div}(B) = 0.$$

Here, $\nabla \times F$ is the curl of F for any vector field on $\mathbb{R}^3$. For example,

$$\nabla \times (\nabla \times F) = \nabla(\operatorname{div} F) - \Delta F,$$

where ΔF is the spatial vector Laplacian; in Cartesian coordinates, it's the Laplacian of each coordinate function of F.

Then since $\operatorname{div}(E) = \operatorname{div}(B) = 0$,

$$E_{tt} = c\nabla \times B_t = c\nabla \times (-c\nabla \times E) = c^2 \Delta E$$

and

$$B_{tt} = -c\nabla \times E_t = -c\nabla \times (c\nabla \times B) = c^2 \Delta B.$$

Hence, each electric and magnetic component of the Maxwell equations satisfy the wave equation, but this time in three spatial dimensions. We will prove that the conservation of energy and a stronger form of the principle of causality hold in $\mathbb{R}^4$. Electric and magnetic signals travel exactly at speed c in a vacuum.

We say a function u of four variables (x, y, z, t) satisfies the wave equation if

$$u_{tt} = c^2 \Delta u$$

with $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplacian on $\mathbb{R}^3$. We similarly define the energy of u by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3} u_t^2 + c^2 |\nabla u|^2 dx dy dz.$$

∇u is the spatial gradient of u and $|\nabla u|$ is the spatial norm of ∇u.

Notice $\frac{1}{2}|\nabla u|_t^2 = \nabla u_t \cdot \nabla u = \operatorname{div}(u_t \nabla u) - u_t \Delta u$.

If u is a solution of the wave equation which vanishes as $r^2 + y^2 + z^2 \to \infty$, then

$$\frac{d}{dt} E(u) = \int_{\mathbb{R}^3} u_t u_{tt} + c^2 \nabla u_t \cdot \nabla u = \int_{\mathbb{R}^3} u_t u_{tt} - c^2 u_t \Delta u + c^2 \operatorname{div}(u_t \nabla u)$$
$$= 0.$$

Waves in Space If $u_{tt} = c^2 \Delta u$ as a function on $\mathbb{R}^4$ and $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for $x \in \mathbb{R}^3$, then

$$u(x_0, t_0) = \frac{1}{4\pi c^2 t_0} \int_S \psi \, dS + \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \int_S \phi \, dS \right].$$

Here, S is the boundary of the ball of center x_0 with radius ct_0 in $\mathbb{R}^3$. This is called **Kirchhoff's formula**. This formula implies $u(x_0, t_0)$ depends on ϕ and ψ on the sphere

$$\{x \in \mathbb{R}^3 \mid |x - x_0| = ct_0\}$$

rather than the entire ball. Conversely, if $x_0 \in \mathbb{R}^3$, then $\phi(x_0)$ and $\psi(x_0)$ only influence u along the boundary

$$\{(x,t) \in \mathbb{R}^4 \mid t > 0, |x - x_0| = ct\}$$

of the cone

$$\{(x,t) \in \mathbb{R}^4 \mid t > 0, |x - x_0| < ct\}.$$

To establish Kirchhoff's formula, we follow [Strauss, '92 pg 223-225] and use the method of spherical means. Let $\overline{u}$ be the average of u on a spatial sphere centered at the origin.

$$\overline{u}(r,t) = \frac{1}{4\pi r^2} \int_{|x|=r} u(x,t) \, dS = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \varphi, \theta, t) \sin(\varphi) d\varphi \, d\theta.$$

If $u_{tt} = c^2 \Delta u$ on $\mathbb{R}^4$, then

$$\overline{u}_{tt} = c^2 \overline{u}_{rr} + 2c^2 \frac{1}{r} \overline{u}_r.$$

This amounts to writing the Laplacian in spherical coordinates and using the rotational invariance of Δ. Then we define

$$v = r\overline{u}.$$

Then

$$v_{tt} = c^2 v_{rr}$$

but only when $r \geq 0$. $v(0,t) = 0$. Also, $v(r,0) = r\overline{\phi}(r)$ and $v_t(r,0) = r\overline{\psi}(r)$.

Therefore,

$$v(r,t) = \frac{1}{2c} \int_{ct-r}^{ct+r} s\overline{\psi}(s) ds + \frac{\partial}{\partial t} \left[\frac{1}{2c} \int_{ct-r}^{ct+r} s\overline{\phi}(s) ds \right]$$

if $0 \leq r \leq ct$.

$$u(0,t) = \lim_{r \to 0} \overline{u}(r,t) = \lim_{r \to 0} \frac{v(r,t)}{r} = \frac{\partial v}{\partial r}(0,t).$$

We compute

$$\frac{\partial}{\partial r}\frac{1}{2c}\int_{ct-r}^{ct+r} s\overline{\psi}(s)ds = \frac{1}{2c}\left((ct+r)\overline{\psi}(ct+r) + (ct-r)\overline{\psi}(ct-r)\right).$$

And

$$\frac{\partial}{\partial r}\frac{1}{2c}\int_{ct-r}^{ct+r} s\overline{\phi}(s)ds = \frac{1}{2c}\left((ct+r)\overline{\phi}(ct+r) + (ct-r)\overline{\phi}(ct-r)\right).$$

Hence,

$$u(0,t) = t\overline{\psi}(ct) + \frac{\partial}{\partial t}t\overline{\phi}(ct) = \frac{1}{4\pi c^2 t}\int_{|x|=ct}\psi(x)dS + \frac{\partial}{\partial t}\frac{1}{4\pi c^2 t}\int_{|x|=ct}\phi(x)dS.$$

A translation argument now implies Kirchhoff's formula. If $w(x,t) = u(x+x_0,t)$, then w still solves the wave equation, so we can apply the above argument to $w(0,t) = u(x_0,t)$.

Now consider $u_{tt} = \Delta u$ in $\mathbb{R}^3$. That is, u solves the wave equation in two spatial dimensions. Then u solves the wave equation in three spatial dimensions. If $u(x,y,0) = 0$ and $u_t(x,y,0) = \psi(x,y)$, then

$$u(0,0,t) = \frac{1}{4\pi c^2 t}\int_{r^2+y^2+z^2=c^2t^2}\psi(x,y)dS.$$

We choose coordinates for S by noting this integral is twice the integral over the hemisphere $z = \sqrt{c^2t^2 - r^2 - y^2}$ in $\mathbb{R}^3$, which is a graph over the disk

$$\{(x,y) \mid r^2 + y^2 \leq c^2t^2\}$$

in the xy-plane. So, the area element becomes

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}dxdy = \frac{c^2t^2}{c^2t^2 - r^2 - y^2}dxdy.$$

Hence,

$$u(0,0,t) = \frac{1}{2\pi c}\int_{r^2+y^2\leq c^2t^2}\frac{\psi(x,y)}{\sqrt{c^2t^2 - r^2 - y^2}}dxdy.$$

In general, if $u(x,y,0) = \phi(x,y)$, then

$$u(x_0,y_0,t_0) = \frac{1}{2\pi c}\int_D \frac{\psi(x_0,y_0)}{\sqrt{c^2t_0^2 - (x-x_0)^2 - (y-y_0)^2}}dxdy + \frac{\partial}{\partial t_0}\left(\frac{1}{2\pi c}\int_D \frac{\phi(x_0,y_0)}{\sqrt{c^2t_0^2 - (x-x_0)^2 - (y-y_0)^2}}dxdy\right)$$

with $D = \{(x,y) \in \mathbb{R}^n \mid (x-x_0)^2 + (y-y_0)^2 \leq c^2t_0^2\}$. This formula implies $u(x_0,y_0,t_0)$ depends on ϕ and ψ in the disk D. Conversely, if $(x_0,y_0) \in \mathbb{R}^n$, then $\phi(x_0,y_0)$ and $\psi(x_0,y_0)$ influence u in the cone

$$\{(x,y,t) \in \mathbb{R}^3 \mid t > 0, (x-x_0)^2 + (y-y_0)^2 \leq c^2t^2\}.$$

https://www.mathpages.com/home/kmath242/kmath242.htm

Waves on a half line Consider the problem

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & \text{if } (x,t) \in (0,\infty)\times(0,\infty) \\ u(0,t) = 0 & \text{if } t \in [0,\infty) \\ u(x,0) = \phi(x) & \text{if } x \in (0,\infty) \\ u_t(x,0) = \psi(x) & \text{if } x \in (0,\infty). \end{cases}$$

We define

$$u_{odd}(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in [0,\infty)\times[0,\infty) \\ -u(-x,t) & \text{if } (x,t) \in (-\infty,0)\times[0,\infty) \end{cases}$$

and note that u_{odd} is odd in x, and is called the odd extension of u.

Moreover,

$$\begin{cases} (u_{odd})_{tt}(x,t) = c^2 (u_{odd})_{xx}(x,t) & \text{if } (x,t) \in \mathbb{R}\times(0,\infty) \\ (u_{odd})(x,0) = (\phi_{odd})(x) & \text{if } x \in \mathbb{R} \\ (u_{odd})_t(x,0) = (\psi_{odd})(x) & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Therefore, from d'Alembert's formula,

$$u_{odd}(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)ds + \frac{1}{2}\left(\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right).$$

Then

$$u(x,t) = \begin{cases} \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) & \text{if } 0 < ct \le x \\ \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)ds + \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) & \text{if } 0 < x < ct. \end{cases}$$

This last line is what we used in the previous lecture.

We now reconcile the solution we obtained for the wave equation on the half line with inhomogenous boundary data using the Laplace transform in lecture 24. If

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & \text{if } (x,t) \in (0,\infty)\times(0,\infty) \\ u(0,t) = f(t) & \text{if } t \in (0,\infty). \\ u(x,0) = \phi(x) & \text{if } x \in (0,\infty) \\ u_t(x,0) = \psi(x) & \text{if } x \in (0,\infty), \end{cases}$$

$$d(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right),$$

and

$$k(x,t) = \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)ds + \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + f\left(t - \frac{x}{c}\right),$$

then

$$u(x,t) = \begin{cases} d(x,t) & \text{if } 0 < ct \le x \\ k(x,t) & \text{if } 0 < x < ct. \end{cases}$$

A Linear Algebra

Definition A.1.

- The symbol $\in$ denotes membership of a set. $\emptyset$ denotes the set with no members. Hence, for all x, $x \notin \emptyset$. If we write $A \subset B$ for two sets A, B, we mean A is a subset of B, and if $A \subseteq B$, then A is either a subset of B or A is equal to B, denoted $A = B$.

- The set of all subsets of a set A is denoted by $\mathcal{P}(A)$, the **power set** of A.

- If A is a set, and for every $x \in A$, $P(x)$ is a statement, then we form the set

$$\{x \in A \mid P(x)\},$$

  which is the set of all $x \in A$ such that $P(x)$.

- The relative complement of two sets A with respect to B, is the set

$$\{x \in A \mid x \notin B\},$$

  denoted by $A \setminus B$.

- If Z is a set of sets, the **union** of Z, denoted by $\bigcup Z$, is the set of members of members of Z. The familiar notation $A \cup B$ for sets A, B is then $A \cup B = \bigcup\{A, B\}$.

- If Z is a set of sets, the **intersection** of Z, denoted $\bigcap Z$, is the set of members of $\bigcup Z$ which are members of each member of Z: $\bigcap Z = \{x \in \bigcup Z \mid \text{ for all } A \in Z, x \in A\}$. The familiar notation $A \cap B$ is then $A \cap B = \bigcap\{A, B\}$.

- If f is a function with domain A and $x \in A$, we write $f(x)$, a member of its codomain, to denote the value of f at x. A function is completely determined by a specification of its domain, codomain, and its value at each member of its domain. Sometimes we refer to the codomain of a function f as codomain(f), and f's domain as domain(f).

- The set of all functions with domain A and codomain B is denoted by Fun(A, B). We sometimes write $f : A \to B$ to mean $f \in \text{Fun}(A, B)$.

- If A is a set and f is a function, we denote the **restriction** of f to A by $f|_A$. If A is not a subset of domain(f), then $f|_A$ is empty.

- If A is a set and f is a function, we denote the **inverse image** of A under f by $f^{-1}(A)$. If A is disjoint from the range of f, then $f^{-1}(A)$ is empty.

- If A is a set and f is a function, we denote the **image** of f under A by $f(A)$. For example, $f(\text{domain}(f)) =$ range(f), the **range** of f.

- If f is an invertible function, we denote the inverse of f by f^{-1}.

- If A is a set, we denote the **identity function** on A by $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$.

- If A, B are sets, A is a subset of B, then the **inclusion** of A into B is the function $\iota_A^B : A \to B$ such that $\iota_A^B(a) = a$ for all $a \in A$. ι is the Greek letter iota.

- If A is a set and r is a set or number, define the **constant r-function** with domain A to be $\mathrm{const}_A^r : A \to \{r\}$ by $\mathrm{const}_A^r(a) = r$ for all $a \in A$.

- **The rectangular product**. If j is a function, denote the set

$$\left\{ f \in \mathrm{Fun}\left(\mathrm{domain}(j), \bigcup \mathrm{range}(j)\right) \mid \text{ for all } i \in \mathrm{domain}(j), f(i) \in j(i) \right\}$$

  by $\bigtimes_j$.

- If j is a function and B is a subset of domain(j), define the **projection** of $\bigtimes_j$ onto $\bigtimes_{j|_B}$ by $P_j^B : \bigtimes_j \to \bigtimes_{j|_B}$ such that $P_j^B(f) = f|_B$ for all $f \in \bigtimes_j$.

- $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ the commutative semigroup of positive integers, called the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set of nonnegative integers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers. We note $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ and $\mathbb{R}$ and $\mathbb{C}$ are both vector spaces over $\mathbb{R}$. The real part of $z \in \mathbb{C}$ is denoted $\Re(z)$, and its imaginary part is denoted $\Im(z)$.

- $<$ is the standard order relation on $\mathbb{R}$, which makes $\mathbb{R}$ the linear continuum with a countable dense subset and no maximum or minimum element, up to order-isomorphism. $\leq$ means either $<$ or $=$. $>$ is the converse of $<$ and $\geq$ is the converse of $\leq$.

- If $a, b \in \mathbb{R}$ and $a < b$, the closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ is denoted by $[a, b]$. The open interval, replacing $\leq$ with $<$, is denoted by (a, b). Unbounded intervals are formed when replacing a with $-\infty$ or b with ∞. The half-open intervals are written $(a, b]$ and $[a, b)$.

- A set A is **finite** if A is either empty or there exists some $n \in \mathbb{N}$ and a bijection between A and $\{1.n\}$. The **size** of such a finite set is 0 if it's empty or n.

- A set A is **countable** if there exists a bijection between A and a subset of $\mathbb{N}$.

- The minimum of a set of real numbers, A if it exists, is denoted by $\min(A)$. The maximum is denoted by $\max(A)$. If A is a finite set of real numbers, then a minimum and maximum exist.

- The **supremum** or **least upper bound** of a set of real numbers, A, denoted $\sup A$, is the unique number which is an **upper bound** of A in the sense that $\sup A \geq a$ for all $a \in A$ and that, if y is any upper bound of A, then $\sup A \leq y$. The **infimum** or **greatest lower bound** of a A, denoted $\inf A$, is the unique number which is a **lower bound** of A in the sense that $\inf A \leq a$ for all $a \in A$ and that, if y is any lower bound of A, then $\inf A \geq y$. By construction or by assumption, the supremum of every nonempty set of real numbers which possesses an upper bound exists. If A is not empty but does not posses an upper bound, then we define $\sup A = \infty$. If A is not empty but does not possess a lower bound, then we define $\inf A = -\infty$. We define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

- If n is a natural number and X_i is either a set or a number for all $i \in [1, n] \cap \mathbb{Z}$, we define, recursively, the set

$$\{X_1.X_n\} = \begin{cases} \{X_1\} & \text{if} \quad n = 1 \\ \{X_1.X_{n-1}\} \cup \{X_n\} & \text{if} \quad n > 1. \end{cases}$$

- If n is a natural number, j is a function with domain $\{1.n\}$, and $j(i) = X_i$ for all $i \in \{1.n\}$, then we denote $\bigtimes_j$ by

$$X_1 \times . \times X_n.$$

- If, for all $i \in \{1.n\}$, $X_i = A$ for some set A, we denote $X_1 \times . \times X_n$ by

$$A^n.$$

  $\mathbb{R}^n$ is called n-dimensional Euclidean space. An element $\mathbf{a} \in A^n$ is sometimes called an n-**tuple** of members of A.

- If A and B are sets, sometimes it's convenient to write $f \in A \times B$ as an ordered pair. That is, for $a \in A$ and $b \in B$, we define $(a, b) \in A \times B$ to be the function $f \in A \times B$ such that $f(1) = a$ and $f(2) = b$.

- Rectangular product of functions. If X is a set, $n \in \mathbb{N}$ and g_i is a function for all $i \in \{1.n\}$, domain $g_i = X$, we define $(g_1.g_n)$, a function on X, by $(g_1.g_n)(x)(i) = g_i(x)$ for all $x \in X$, for all $i \in \{1.n\}$. Hence, the codomain of this function is codomain$(g_1) \times . \times$ codomain(g_n).

- If $A \subseteq X_1 \times . \times X_n$, then we consider $A \times B$ as a subset of $X_1 \times . \times X_{n+1}$, $X_{n+1} = B$ with the following identification. $f \in A \times B$ if and only if $g \in X_1 \times . \times X_n \times B$ if $g \in X_1 \times . \times X_{n+1}$ with $f(1)(i) = g(i)$ for all $i \in \{1.n\}$ and $f(2) = g(n+1)$.

- If A is a nonempty finite set of integers, V a vector space over $\mathbb{C}$ and j is a function with domain A and codomain V, we define the **sum** of $j(A)$, denoted $\sum_{n \in A} j(n)$, recursively by

$$\begin{cases} \sum_{n \in A} j(n) = j(\max(A)) & \text{if} \quad A = \{\max(A)\} \\ \sum_{n \in A} j(n) = j(\max(A)) + \sum_{n \in A \setminus \{\max(A)\}} j|_{A \setminus \{\max(A)\}}(n) & \text{if} \quad A \setminus \{\max(A)\} \neq \emptyset \end{cases}$$

  Then the familiar notation $\sum_{i=0}^{n} a_i$ is $\sum_{i \in \{0.n\}} j(i)$ if $j(i) = a_i$ for all $i \in \{0.n\}$.

Exercise A.2. We have the following change of index for the sum. $\sigma : B \to A$ is a bijection between two finite sets of integers, A, B, V a vector space over $\mathbb{C}$, j is a function with domain B and codomain V, then

$$\sum_{m \in A} j(m) = \sum_{n \in B} j \circ \sigma(n).$$

The can be proved by induction on the size of A.

Definition A.3.

- A map, $L : V \to W$ between two vector spaces over $\mathbb{C}$, V and W, is **linear** if, whenever $a, b \in \mathbb{R}$ and $u, v \in V$, then
$$L(au + bv) = aL(u) + bL(v).$$
- The **kernel** of a linear map, $L : V \to W$, is the set
$$\ker(L) = \{v \in V \mid L(v) = 0\}.$$
- If V, W are vector spaces, the collection of all linear maps with domain V and codomain W is $\mathcal{L}(V, W)$.
- A **linear isomorphism** is a injective and surjective linear map.

Exercise A.4.

- If $L \in \mathcal{L}(V, W)$ between vector spaces V, W, then $L(0) = 0$.
- If V is a vector space over $\mathbb{C}$ with addition $+$ and scalar multiplication $\cdot$ and $A \subset V$, then A is a vector space with addition $+|_A$ and scalar multiplication $\cdot|_A$ if and only if the images of $+|_A$ and $\cdot|_A$ are subsets of A. In this case, we say A is a **subspace** of V.
- The kernel of a linear map is a subspace of its domain.
- The real and imaginary parts of $\mathbb{C}^n$, denoted $\mathfrak{R}, \mathfrak{I}$, defined by $\mathfrak{R}(\mathbf{z})(i) = \mathfrak{R}(\mathbf{z}(i))$ and $\mathfrak{I}(\mathbf{z})(i) = \mathfrak{I}(\mathbf{z}(i))$ for all $\mathbf{z} \in \mathbb{C}^n$, $i \in \{1.n\}$, are linear maps onto $\mathbb{R}^n$.
- The image of a linear map is a subspace of its codomain. In particular, there is a linear isomorphism between the real part of $\mathbb{C}^n$, $\mathfrak{R}(\mathbb{C}^n)$ and $\mathbb{R}^n$.
- If A is a set and V is a vector space over $\mathbb{C}$, then the set of all function with domain A and codomain V, $\mathrm{Fun}(A, V)$, is a vector space with function addition defined by, for all $f, g \in \mathrm{Fun}(A, V)$, $f + g : A \to V$, $(f + g)(a) = f(a) + g(a)$ for all $a \in A$, and function scalar multiplication defined by, for all $f \in \mathrm{Fun}(A, V)$, for all $r \in \mathbb{C}$, $(rf) : A \to V$, $(rf)(a) = r \cdot f(a)$ for all $a \in A$.
- If V and W are vector spaces, $\mathcal{L}(V, W)$ is a vector space with the function addition and scalar multiplication.
- If n is a natural number, V_i is a vector space over $\mathbb{C}$ for all $i \in \{1.n\}$, then $V_1 \times . \times V_n$ is a vector space over $\mathbb{C}$ with function addition and scalar multiplication.

Definition A.5.

(a) If V is a vector space over $\mathbb{C}$, the **dual space** of V is the set of all **linear functionals**, denoted
$$V^* = \mathcal{L}(V, \mathbb{C}).$$

(b) If A is a set, an **equivalence relation** on A is a relation on A, R, which is a subset of $A \times A$ with the properties: For all $x, y, z \in A$,

Symmetry: If $(x, y) \in R$, then $(x, y) \in R$.

Reflexivity: $(x, x) \in R$.

Transitivity: If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

If $(x, y) \in R$, we say x and y are equivalent.

(c) If A is a set, R is an equivalence relation on A, and $a \in A$, then the **equivalence class** of a with respect to R is the set of all $x \in A$ equivalent to a, denoted by $[a] = \{x \in A \mid (x, a) \in R\}$. Elements of an equivalence class are called **representatives** of it. Two representatives are in the same equivalence class if and only if they're related to each other under the equivalence relation. The set of all equivalence classes of A with respect to R is called the **quotient** of A with respect to R, denoted A/R, and is a subset of $\mathcal{P}(A)$.

Exercise A.6.

(a) The dual space of a vector space is itself a vector space with function addition and function scalar multiplication.

(b) If V is a vector space and V is finite dimensional, V and its dual V^* are isomorphic.

(c) If V is a vector space, its **double dual**, V^{**}, which is the dual of V^*, is isomorphic to V. The following is an isomorphism. Define $\Phi : V \to V^{**}$ by $\Phi(v)(f) = f(v)$, for all $v \in V$, for all $f \in V^*$.

(d) The relation $\leq$ on $\mathbb{R}$ is reflexive and transitive, but not symmetric.

(e) If A is a set, the relation R on A given by $(x, y) \in R$ if and only if $x = y$ is the only equivalence relation which is also **anti-symmetric**: if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

(f) If V is a vector space and R is an equivalence relation on V, then the **quotient** V/R is a vector space with the following addition and scalar multiplication operators. If $[x], [y] \in V/R$, then $[x]+[y] = [x+y]$. If $[x] \in V/R$, and $r \in \mathbb{R}$, then $r \cdot [x] = [r \cdot x]$. In particular, these operators are functions on $V/R \times V/R$ and $\mathbb{C} \times V/R$, respectively.

(g) If $W \subseteq V$ is a subspace of a vector space V, then the relation on V defined by $x - y \in W$ for all $(x, y) \in V \times V$ is an equivalence relation. We refer to the quotient of V with respect to this relation by V/W.

Definition A.7. If n is a natural number, we define the **standard basis** on $\mathbb{C}^n$ as follows. We define, for all $i \in \{1.n\}$,

$$\mathbf{e}_i^n \in \mathbb{C}^n$$

such that

$$\mathbf{e}_i^n(j) = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \in \{1.n\} \setminus \{i\} \end{cases}$$

If $k \in \mathbb{N}$ and $k \leq n$, identify $\mathbb{C}^k$ as a subspace of $\mathbb{C}^n$, since $\mathbf{e}_i^k = \mathbf{e}_i^n|_{\{1.k\}}$ for all $i \in \{1.k\}$. Hence, we refer to $\mathbf{e}_i^n$ by $\mathbf{e}_i$ without confusion.

Definition A.8. For each $i = 1.n$, define the real-valued i**th coordinate function of** $\mathbb{C}^n$, r^i by

$$r^i(\mathbf{x}) = \mathbf{x}(i)$$

for all $\mathbf{x} \in \mathbb{C}^n$.

Exercise A.9. If k is a natural number less than or equal to a natural number n, and $I \subseteq \{1.n\}$ such that I has size k, let ι_I be the inclusion of $\bigtimes_j$ into $\mathbb{C}^n$ and P^I be the projection of $\mathbb{C}^n$ onto $\bigtimes_j$ with $j(i) = \mathbb{C}$ for all $i \in I$. Denote $\bigtimes_j$ by $\mathbb{C}_n^I$.

(a) $\mathbb{C}_n^I$ is isomorphic to $\mathbb{C}^k$.

(b) ι_I is an injective linear map and

$$\iota_I(\mathbb{C}_n^I) = \bigcap \{\ker(r^i) \in \mathcal{P}(\mathbb{C}^n) \mid i \in \{1.n\} \setminus I\} = \left(P^{\{1.n\}\setminus I}\right)^{-1}(\{0\}).$$

(c) $P^{\{k\}} = r^k$.

(d) P^I is a surjective linear map and

$$P^I\left(\left(P^{\{1.n\}\setminus I}\right)^{-1}(\{0\})\right) = \mathbb{C}_n^I.$$

(e) $\iota_I \circ P^I = \sum_{i\in I} r^i \mathbf{e}_i$ as functions on $\mathbb{C}^n$.

(f) $P^I \circ \iota_I = \sum_{i\in I} r^i \mathbf{e}_i$ as functions on $\mathbb{C}_n^I$.

Definition A.10. A **complex inner product** on a complex vector space V is a complex-valued function, $\star$, defined on $V \times V$, with evaluation written with infix notation, satisfying

1. Linearity in the first argument: For any $a, b \in \mathbb{C}$, for any $x, y, z \in V$,

$$(ax + by) \star z = a(x \star z) + b(y \star z).$$

2. Conjugate Symmetry: For any $x, y \in V$,

$$x \star y = \overline{y \star x}.$$

If $a \in \mathbb{C}$, $\overline{a}$ denotes its complex conjugate.

3. Positive-definiteness: For any $x \in V \setminus \{0\}$,

$$x \star x > 0.$$

For all $x \in V$, denote $\sqrt{x \star x}$ by $|x|$, called the **norm** of x induced by $\star$.

We say the pair $(V, \star)$ is an **inner product space**.

Definition A.11. If we define, for every $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} \mathbf{x}(i)\overline{\mathbf{y}(i)},$$

it follows $\cdot$ is an inner product on $\mathbb{C}^n$, called the **standard inner product** on $\mathbb{C}^n$. Two nonzero vectors, $\mathbf{x}, \mathbf{y}$ are **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

Exercise A.12. The standard basis on $\mathbb{C}^n$ form an **orthonormal basis**, meaning $\mathbf{e}_i$ and $\mathbf{e}_j$ are orthgonal if $i \neq j$ and $|\mathbf{e}_i| = 1$ for all $i \in \{1.n\}$.

Exercise A.13. Suppose $(V, \star)$ is a complex inner product space.

- A real inner product is linear in the second argument. (So we say an inner product is **bilinear**.)
- $0 \star x = 0$ for all $x \in V$.
- $|x| = 0$ if and only if $x = 0$.
- $|x \star y| \leq |x||y|$ for all $x, y \in V$. This is called the **Cauchy-Schwarz inequality**. $|x \star y|$ is the norm of $x \star y$ induced by the standard inner product on $\mathbb{C}$. And $|x|$ is the norm of x induced by $\star$. **Hint:** First prove this inequality when $y = 0$. Then if $y \neq 0$, define $u = x - \frac{x \star y}{|y|^2} \cdot y$ and consider $(u \star u)^2$.
- $|x + y| \leq |x| + |y|$ for all $x, y \in V$. This is called the **triangle inequality**. A norm induced by an inner product is said to be **subadditive**.

Definition A.14. We suppose V is a vector space over $\mathbb{C}$.

(a) A function $p \in \mathrm{Fun}(V, \mathbb{R})$ is

(i) **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.
(ii) **absolutely homogeneous** if $p(ax) = |a|p(x)$ for all $a \in \mathbb{C}$, $x \in V$.
(iii) **positive definite** if, for all $x \in V$, if $p(x) = 0$, then $x = 0$.

(b) A **seminorm** on V is a subadditive, absolutely homogeneous function on V.

(c) A **norm** on V is a positive definite seminorm.

Exercise A.15.

(a) If $(V, \star)$ is an inner product space, then the norm induced by $\star$ is a norm on V.

(b) A norm, p, on a complex vector space V is induced by a complex inner product if and only if

$$(p(x + y))^2 + (p(x - y))^2 = 2(p(x))^2 + 2(p(y))^2$$

for all $x, y \in V$. This is called the parallelogram law. In this case, the inner product, $\star$, inducing p, is

$$x \star y = \frac{(p(x + y))^2 - (p(x - y))^2}{4}$$

for all $x, y \in V$.

A.1 Tensors

Definition A.16. If V and W are vector spaces, then we define $V^* \otimes W$ to be the set $\mathcal{L}(V, W)$, if V^* is the dual space of V. If $f \in V^*, w \in W$, we define $f \otimes w \in V^* \otimes W$ by $f^* \otimes w(v) = f^*(v)w$ for all $v \in V$.

Definition A.17. If V and W are vector spaces, then we define $V^* \otimes W$ to be the set $\mathcal{L}(V, W)$, if V^* is the dual space of V. If $f \in V^*, w \in W$, we define $f \otimes w \in V^* \otimes W$ by $f^* \otimes w(v) = f^*(v)w$ for all $v \in V$.

Definition A.18. If n is a natural number, V, W are vector spaces over $\mathbb{C}$, then $\mathcal{L}^n(V, W)$ is the subspace of all $f \in \mathrm{Fun}(V^n, W)$ such that, for all $\mathbf{v} \in V^n$, if $P^{\{j\}} \circ I^i_{\mathbf{v}}(v) = P^{\{j\}}(\mathbf{v})$ if $j \neq i$ and $P^{\{j\}} \circ I^i_{\mathbf{v}}(v) = v$ if $i = j$, then $f \circ I^i_{\mathbf{v}} \in \mathcal{L}(V, W)$ for all $i \in \{1.n\}$.

Theorem A.19. If V, W are vector spaces, and $m, n \in \mathbb{N}$, then there exists a linear isomorphism between $\mathcal{L}^{m+n}(V, W)$ and $\mathcal{L}^m(V, \mathcal{L}^n(V, W))$.

Proof. We define such an isomorphism. Define $\Phi : \mathcal{L}^m(V, \mathcal{L}^n(V, W)) \to \mathcal{L}^{m+n}(V, W)$ by $\Phi(f)(\mathbf{v}) = f(P^{\{1.m\}}(\mathbf{v}))(P^{\{m+1.m+n\}}(\mathbf{v}))$ for all $\mathbf{v} \in V^{m+n}$. Φ is linear by the definition of function addition. If $\Phi(f) = 0$, then $f = 0$. So, Φ is injective. If $g \in \mathcal{L}^{m+n}(V, W)$, define $f \in L^m(V, \mathcal{L}^n(V, W))$ by $f(v)(v') = g(v'')$ with $P^{\{1.m\}}(v'') = v$ and $P^{\{m+1.n+m\}}(v'') = v'$. Then $\Phi(f) = g$. Hence, Φ is surjective. □

B Topology

Definition B.1. A set T together with a set of subsets of T, $\tau \subseteq \mathcal{P}(T)$ is called a **topology** on T if the following holds.

- $\emptyset, T \in \tau$.
- If $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$.
- If $\mathcal{U} \subseteq \tau$ and $\mathcal{U}$ is finite, then $\bigcap \mathcal{U} \in \tau$.

If the pair (T, τ) is a topology and $U \in \tau$, then U is called **open** in τ, in (T, τ) or just in T if the context is clear. If the pair (T, τ) is a topology, we call T a **topological space** and sometimes refer to τ as the topology itself. If U is open, then $T \setminus U$ is **closed**. A **neighborhood** of x in a topological space T is a set V such that $x \in V$ and there exists $U \subseteq V$ such that U is open and $x \in U$.

Definition B.2.

- If (X, τ) is a topology and (Y, υ) is a topology, a function $f : X \to Y$ is **continuous** on X with respect to τ if, for every $U \in \upsilon$, the pre-image $f^{-1}(U) \in \tau$. Denote the set of all continuous functions with domain X and codomain Y by $C(X, Y)$. We denote $C(X, \mathbb{C}) = C(X)$ if $\mathbb{C}$ is the standard topology.
- A **homeomorphism** is a bijective continuous function between two topological spaces such that its inverse is also continuous.
- A function between two topological spaces is **open** if the image of every open set in its domain is open in its codomain.

Exercise B.3.

(a) The composition of two continuous functions is continuous.

(b) Every function whose domain is the discrete topology is continuous.

(c) There exists a continuous bijection whose inverse is not continuous.

(d) Any homeomorphism is an open function.

(e) If X, Y are topological spaces, then $f \in C(X, Y)$ if and only if, for every $x \in X$, for any neighborhood, V, of $f(x)$, there is some neighborhood, U, of x such that $f(U) \subseteq V$.

Definition B.4.

(a) A **base** of a topology (X, τ) is any $B \subseteq \tau$ such that, if $U \in \tau$, then there is some $C \subseteq B$ such that

$$U = \bigcup C.$$

(b) A **subbase** of a topology (X, τ) is any $S \subseteq \tau$ such that, if

$$B = \left\{ \bigcap F \in \tau \mid F \subseteq S \text{ and } F \text{ is finite} \right\},$$

then B is a base of (X, τ).

(c) Suppose X is a set and F is a set of functions such that if $f \in F$, then $f \in \text{Fun}(X, \text{codomain}(f))$ with codomain(f) some topological space. The **initial topology** on X with respect to F is the smallest (by way of the subset relation) set of open sets such that each $f \in F$ is continuous. In other words, the initial topology is the intersection of all topologies which every $f \in F$ is continuous on.

(d) Suppose X is a set and F is a set of functions such that if $f \in F$, then $f \in \text{Fun}(\text{domain}(f), X)$ with domain(f) some topological space. The **final topology** on X with respect to F is the largest (by way of the subset relation) set of open sets such that each $f \in F$ is continuous. In other words, the final topology is the union of all topologies which every $f \in F$ is continuous on.

Exercise B.5.

(a) A set $B \subseteq \mathcal{P}(X)$ is a base of a topology if and only if

$$\bigcup B = X$$

and, for every $B_1, B_2 \in B$, if $x \in B_1 \cap B_2$, there is some $B_3 \in B$ such that

$$x \in B_3 \subseteq B_1 \cap B_2.$$

(b) A set S is a subbase of a topology (X, τ) if and only if

$$\tau = \bigcap \{\kappa \subseteq \mathcal{P}(X) \mid (X, \kappa) \text{ is a topology and } S \subseteq \kappa\}.$$

In other words, the topology which contains S as a subbase is the intersection of all topologies containing S.

(c) (X, τ) is the initial topology with respect to a collection of functions F if and only if

$$\{f^{-1}(U) \in \mathcal{P}(X) \mid f \in F \text{ and } U \subseteq \text{codomain}(f) \text{ is open}\}$$

is a subbase of (X, τ).

(d) (X, τ) is the final topology with respect to a collection of functions F if and only if $U \in \tau$ if and only if $f^{-1}(U)$ is open in domain(f) for every $f \in F$.

Exercise B.6.

(a) If X is a set and $\tau = \mathcal{P}(X)$, then (X, τ) is a topology, called the **discrete topology** on X.

(b) If X is a set and $\tau = \{\emptyset, X\}$, then (X, τ) is a topology, called the **trivial topology** on X.

(c) If $n \in \mathbb{N}$, $E \subseteq \mathbb{C}^n$, the **standard** topology on E is the topology with subbase the collection of all subsets of the form $E \cap B_r(x)$, with $r > 0$ and $x \in E$, $B_r(x)$ a **disk of radius r, center** x, defined as the set

$$\{|y - x| < r \mid y \in \mathbb{C}^n\}.$$

In fact, this collection is a base of this topology. And $|y - x|$ is the norm induced by the standard inner product on $\mathbb{C}^n$.

(d) If (X, τ) is a topological space, $A \subseteq X$, the **subspace topology** on A with respect to (X, τ) is the initial topology on A with respect to the inclusion map ι_A^X from A into X.

- The subspace topology on A with respect to (X, τ) is the pair (A, τ_A), with

$$\tau_A = \{U \cap A \mid U \in \tau\}.$$

(e) Suppose j is a function such that its range consists of topological spaces. Then there are many topologies with $\bigtimes_j$ its topological space.

(i) Define

$$B = \left\{ \bigtimes_k \in \mathcal{P}\left(\bigtimes_j\right) \middle| \; k(i) \text{ is open in } j(i) \text{ for all } i \in \text{domain}(j) \right\}.$$

Then B is a base of a topology, called the **box topology** on $\bigtimes_j$.

(ii) Define, for each $i \in \text{domain}(j)$, the projection $P^{\{i\}} \in \text{Fun}(\bigtimes_j, j(i))$ such that $P^{\{i\}}(f) = f(i)$ for all $f \in \bigtimes_j$. The initial topology on $\bigtimes_j$ with respect to $\{P^{\{i\}} \in \text{Fun}(\bigtimes_j, j(i)) \mid i \in \text{domain}(j)\}$ is called the **product topology** on $\bigtimes_j$.

(1) Each projection defined above is open in the product topology.

(2) A base of the product topology is the collection of all sets of the form $\bigtimes_k$ such that $k(i)$ is open in $j(i)$ for all $i \in \text{domain}(j)$ and $k(i) = j(i)$ for all but finitely many $i \in \text{domain}(j)$.

(iii) The box topology contains the product topology, and the open sets coincide when the domain of j is finite.

(iv) The standard topology on $\mathbb{C}^n$ is the product (box) topology on $\mathbb{C}^n$.

(v) The product topology on $\mathbb{R}^n$ is the standard topology on $\mathbb{C}^n$, which is the subspace topology on $\mathbb{R}^n$ with respect to the standard topology on $\mathbb{C}^n$.

Definition B.7. If (X, τ) is a topology and $S \subseteq X$,

- The **closure** of S relative to X is the smallest closed set containing S, denoted $\text{Cl}(S)$. Namely,

$$\text{Cl}(S) = \bigcap\{C \in \mathcal{P}(X) \mid C \text{ is closed and } S \subseteq C\}.$$

- The **interior** of S relative to X is the largest open set contained in S denoted $\text{Int}\, S$. Namely,

$$\text{Int}\, S = \bigcup\{U \in \tau \mid U \text{ is open and } U \subseteq S\}.$$

- The boundary of S relative to X is $\text{bdy}\, S$. Namely, $\text{bdy}\, S = \text{Cl}(S) \setminus \text{Int}\, S$.

Exercise B.8. If (X, τ) is a topology and $S \subseteq X$, then $x \in \operatorname{bdy} S$ if and only if, for all $T \in \tau$, if $x \in T$, then $T \cap S \neq \emptyset$ and $T \setminus S \neq \emptyset$.

Exercise B.9. If X is a topological space,

(a) And $S \subseteq X$, then $\operatorname{bdy}(X \setminus S) = \operatorname{bdy} S$.

(b) and $S \subseteq X$, then $\operatorname{bdy} \operatorname{Cl}(S) \subseteq \operatorname{bdy} S$, $\operatorname{Int} S$ is open, and $\operatorname{Cl}(S)$ is closed.

(c) $U \subseteq X$ and U is open, then $\operatorname{bdy} U = \operatorname{bdy} \operatorname{Cl}(U)$ and $\operatorname{Int} U = U$.

(d) and C is closed, then $\operatorname{Cl}(C) = C$.

Definition B.10.

(a) A topological space X is **separated** if, for every $x \in X$,

$$\{x\} = \bigcap \{V \mid V \text{ is a closed neighborhood of } x\}.$$

(b) A topology (X, τ) is **second countable** if it has a countable base.

(c) A topological space is **connected** if it is not the disjoint union of two open sets.

Exercise B.11.

(a) A topological space X is separated if and only if, for every $x, y \in X$, if $x \neq y$, then there exist neighborhoods V of x and U of y such that $V \cap U = \emptyset$.

(b) If $n \in \mathbb{N}$, and $E \subseteq \mathbb{C}^n$, then E with respect to the standard topology is separated and second countable.

(c) A topological space is a union of disjoint open connected sets, called its **connected components**.

(d) If Y is a topological space, X, U are nonempty open subsets of Y, $\operatorname{bdy} U \neq \emptyset$, X is connected, $X \cap U \neq \emptyset$ and $X \setminus U \neq \emptyset$, then $X \cap \operatorname{bdy} U \neq \emptyset$.

Definition B.12.

(a) A **sequence** of a set, X, is any $s \in \operatorname{Fun}(\mathbb{N}, X)$.

(b) A **subsequence** of a sequence s of a set X is any sequence t such that $t = s \circ b$ with $b \in \operatorname{Fun}(\mathbb{N}, \mathbb{N})$ an increasing function in that $b(m) < b(n)$ for all $m < n$.

(c) A sequence s of a topological space X **converges** to $x \in X$ if, for every neighborhood N or x, there exists some $M \in \mathbb{N}$ such that $s(m) \in N$ for all $m \geq M$.

(d) A set S in a topological space is **sequentially open** if, for every $x \in S$, for every sequence s which converges to x, there is some $M \in \mathbb{N}$ such that $s(m) \in S$ for all $m \geq M$.

(e) A topological space is a **sequential space** if every sequentially open set is open.

(f) A topological space, X, is **first countable** if every $x \in X$ has a countable **neighborhood base**. A neighborhood base of x is a collection of neighborhoods of x, $\mathcal{B}$, such that, for every neighborhood of x, N, there is some $B \in \mathcal{B}$ such that $B \subseteq N$.

Exercise B.13.

(a) If a sequence converges to a point, then any subsequence of that sequence converges to the same point.

Theorem B.14. A first countable space is sequential.

Proof. Suppose X is a first countable space, $U \subseteq X$ and U is sequentially open. Suppose U is not open. That is, suppose there is some $x \in U$ such that, for every open neighborhood, V, of x, there exists some $y \in V$ such that $y \notin U$. Since X is first countable, fix a countable neighborhood base, $\{B(n) \mid n \in \mathbb{N}\}$, of x. Without loss of generality, $B(n+1) \subseteq B(n)$ and $B(n)$ is open for all $n \in \mathbb{N}$. Then, since U is not open, for every $n \in \mathbb{N}$, there exists some $y(n) \in B(n)$ such that $y(n) \notin U$. If $s_x \in \mathrm{Fun}(\mathbb{N}, X)$ is defined by $s_x(n) = y(n)$ for all $n \in \mathbb{N}$, then we claim s_x converges to x in X. This is because, for every neighborhood of x, V, since $\{B(n) \mid n \in \mathbb{N}\}$ is a neighborhood base of x, then there is some $n \in \mathbb{N}$ such that $B(n) \subseteq V$. Hence, $s_x(j) \in V$ for all $j \geq n$ since $B(m+1) \subseteq B(m)$ and $s_x(m) \in B(m)$ for all $m \in \mathbb{N}$. By definition, s_x converges to x. This contradicts the sequential openness of U, since $x \in U$ and $U \setminus s_x(\mathbb{N}) = \emptyset$. □

Exercise B.15. If X is a topological space and $x \in X$ has a countable neighborhood base,

(a) then x has a countable neighborhood base consisting of open sets.

(b) then there exists a countable neighborhood base of x, $\{B(n) \mid n \in \mathbb{N}\}$ such that $B(n+1) \subseteq B(n)$ for all $n \in \mathbb{N}$. *Hint:* Consider finite intersections of elements of the countable neighborhood base of x.

Definition B.16.

(a) If f is a function between topological vector spaces, X and Y, then if $a \in X$, $b \in Y$ we write

$$\lim_a f \to b$$

if, for every sequence s of X, if s converges to a, then $f \circ s$ converges to b.

Exercise B.17. We suppose X and Y are topological spaces.

(a) If X is separated and s is a sequence of X such that s converges to both $a \in X$ and $b \in X$, then $a = b$.

(b) If $f \in C(X, Y)$, then

$$\lim_a f \to f(a)$$

for every $a \in X$.

(c) If X is a sequential space and $\lim_a f \to f(a)$ for every $a \in X$, then $f \in C(X, Y)$.

(d) There are some topological spaces X and Y and function $f \in \mathrm{Fun}(X, Y)$ such that $\lim_a f \to f(a)$ for every $a \in X$ but $f \notin C(X, Y)$.

Theorem B.18. If X is a set and and Y is a topological space, then $\mathrm{Fun}(X, Y)$ is naturally a product space, $\bigtimes_j$, if $j(x) = Y$ for every $x \in X$. The product topology on $\mathrm{Fun}(X, Y)$ is called the topology of **pointwise convergence** because a sequence $s \in \mathrm{Fun}(\mathbb{N}, \mathrm{Fun}(X, Y))$ converges to $f \in \mathrm{Fun}(X, Y)$ if and only if, for every $x \in X$, the sequence s_x defined by $s_x(n) = s(n)(x)$ for every $n \in \mathbb{N}$, converges to $f(x)$.

Proof. We suppose s is a sequence of $\text{Fun}(X, Y)$ which converges to f. We suppose $x \in X$. We define $s_x \in \text{Fun}(\mathbb{N}, Y)$ by $s_x(n) = s(n)(x)$ for every $n \in \mathbb{N}$. Fix an open neighborhood of $f(x)$, say V. If we define

$$k(a) = \begin{cases} V & \text{if} \quad a = x \\ Y & \text{if} \quad a \neq x, \end{cases}$$

then $\bigtimes_k$ is open in $\text{Fun}(X, Y)$ and $f \in \bigtimes_k$. Hence, since s converges to f, there exists some $M \in \mathbb{N}$ such that $s(n) \in \bigtimes_k$ for all $n \geq M$. Hence, $s_x(n) = s(n)(x) \in V$ for all $n \geq M$. Hence, s_x converges to $f(x)$.

Conversely, we suppose s_x converges to $f(x)$ for all $x \in X$. Suppose U is open in $\text{Fun}(X, Y)$, containing f. Then there exists some $\bigtimes_k$, open in $\text{Fun}(X, Y)$, containing f, such that $k(a) = Y$ for all but finitely many $a \in X$. Say, $k|_{X \setminus \{a_1 . a_n\}} = \text{const}_X^Y$ for some $\{a_1 . a_n\} \subseteq X$ and $k(a_i)$ is open in Y for all $i \in \{1.n\}$. Then, for all $i \in \{1.n\}$, there is some $M_i \in \mathbb{N}$ such that $s_{a_i}(n) \in k(a_i)$ for all $n \geq M_i$. If $M = \max\{M_i \mid i \in \{1.n\}\}$, then, for all $x \in X$, $P^{\{x\}}(s(n)) = s(n)(x) = s_x(n) \in k(x)$ for all $n \geq M$. Hence, $s(n) \in \bigtimes_k = \bigcap\{(P^{\{x\}})^{-1}(k(x)) \mid x \in X\}$ for all $n \geq M$. Hence, s converges to f. □

B.1 Compactness

Definition B.19.

(a) If X is a topological space, an **open cover** of X is any $\mathcal{U} \subseteq \mathcal{P}(X)$ such that, for every $U \in \mathcal{U}$, U is open and $X = \bigcup \mathcal{U}$.

(b) A topological space X is **compact** if, for every open cover $\mathcal{U}$ of X, there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is finite and $\mathcal{V}$ is an open cover of X.

(c) A subset A of a topological space X is **compact** if it is compact with respect to the subspace topology A with respect to X.

Exercise B.20.

(a) If X is compact, $C \subseteq X$ and C is closed, then C is compact.

(b) If X is separated, $C \subseteq X$, and C is compact, then C is closed.

(c) There is some non separated space X and compact subset C of X such that C is not closed.

(d) There is some non separated space X and compact subset C of X such that $\text{Cl}(C)$ is not compact.

(e) If $f \in C(X, Y)$, $K \subseteq X$ and K is compact, then $f(K)$ is compact.

(f) A subset, K, of a sequential space, X, is compact if and only if, for every sequence $s \in \text{Fun}(\mathbb{N}, K)$, there exists $x \in K$ and some subsequence $s_1 \in \text{Fun}(\mathbb{N}, K)$ such that s_1 converges to x in X.

Theorem B.21. [Taylor, '11, Prop B.1.1] If the domain of j is finite, and $j(i)$ is a compact sequential space for all $i \in \text{domain}(j)$, then $\bigtimes_j$ is a compact sequential space.

Proof. Let σ be a bijection from $\{1.k\}$ to domain(j), with k the size of domain(j). Fix a sequence $s \in \mathrm{Fun}(\mathbb{N}, \bigtimes_j)$. Then, for every $i \in \{1.k\}$, define $s_i(n) = s(n)(\sigma(i))$ for every $n \in \mathbb{N}$. Then s_i is a sequence of $j(i)$. Hence, since $j(\sigma(1))$ is compact and sequential, there exists $x_1 \in j(\sigma(1))$ and an increasing map b_1 such that $s_1 \circ b_1$ converges to x_1 in $j(\sigma(1))$. If $i \in \{1.k\} \setminus \{1\}$, then there exists $x_i \in j(\sigma(i))$ and an increasing map b_i such that $s_i \circ B_i$ converges to x_i in $j(\sigma(i))$, with $B_i = b_i$ if $i = 1$, and $B_i = B_{i-1} \circ b_i$ if $i > 1$.

Define $x \in \bigtimes_j$ by $x(\sigma(i)) = x_i$ for all $i \in \{1.k\}$ and $p = s \circ B_k$. Then p is a subsequence of s and since every subsequence converges to what the sequence converges to, we claim p converges to x.

Fix an open set U containing x. Fix $i \in$ domain(j). Then the sequence p_i defined by $p_i = s_{\sigma^{-1}(i)} \circ B_k$ converges to x_i. Then since the projection $P^{\{i\}}$ is open in $j(i)$, $P^{\{i\}}(U)$ is open in $j(i)$ and $x_i \in P^{\{i\}}(U)$. Moreover, there is some $M_i \in \mathbb{N}$ such that $p_i(n) \in P^{\{i\}}(U)$ for all $n \geq M_i$.

Since

$$P^{\{i\}}(p(n)) = p(n)(i) = s(B_k(n))(i) = s_{\sigma^{-1}(i)}(B_k(n)) = p_i(n) \in P^{\{i\}}(U),$$

it follows $p(n) \in U$ for all $n \geq M_i$. Hence, p converges to x.

□

B.2 Topological Vector Spaces

Definition B.22.

(a) A **topological vector space** is a vector space V, which is also a topological space such that addition is continuous on $V \times V$ and scalar multiplication is continuous on $\mathbb{C} \times V$ with the product topologies.

(b) A **Cauchy sequence** of a topological vector space V is a sequence s of V such that, for any neighborhood N of 0, there exists some $M \in \mathbb{N}$ such that $s(m) - s(n) \in N$ for all $m, n \geq M$.

(c) A topological vector space is **complete** if every Cauchy sequence converges to some point in the space.

(d) A **Frechét space**, V is a separated, complete topological vector space whose topology is generated by a set of seminorms in the following way. The topology on V is the initial topology on V with respect to a set of seminorms.

There exists some set P such that, for all $p \in P$, p is a seminorm on V. And U is open in V if and only if, for every $y \in U$, there exists some $K \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\{x \in V \mid p(k)(x-y) < \varepsilon \text{ for all } k \in \{1.K\}\} \subseteq U.$$

(e) A **Banach space** is a complete topological vector space whose topology is generated by a norm in the following way. There exists a norm, p such that U is open in V if and only if, for every $y \in U$, there exists some $\varepsilon > 0$ such that

$$\{x \in V \mid p(x-y) < \varepsilon\} \subseteq U.$$

Exercise B.23. If V is a Frechét space, whose topology is generated by a set of seminorms P, and, for every $x \in V$, $p \in P$, $\varepsilon > 0$, $B_\varepsilon(p, x) = \{x \in V \mid p(x-y) < \varepsilon\}$,

(a) then $\{B_\varepsilon(p, x) \mid \varepsilon > 0, p \in P, x \in V\}$ is a subbase of V.

Exercise B.24. If V is a Banach space, whose topology is generated by a norm p, and, for every $x \in V$, $\varepsilon > 0$, the **open disk of radius ε, center** x is defined as

$$\{x \in V \mid p(x - y) < \varepsilon\},$$

denoted

$$B_\varepsilon(x),$$

then

(a) $B_\varepsilon(x)$ is open in V.

(b)
$$\{B_{\frac{1}{n}}(x) \mid n \in \mathbb{N}, x \in V\}$$
is a neighborhood base of x. Hence, any Banach space is first countable.

(c) a sequence s of V converges to $x \in V$ if and only if, for every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $p(s(n) - x) < \varepsilon$ for every $n \geq M$.

(d) if $n \in \mathbb{N}$, $\mathbb{C}^n$ with the standard topology is a Banach space with respect to the norm induced by the standard inner product.

B.2.1 Uniform Convergence

Definition B.25. Suppose Y is a topological vector space, X is a set, $\mathcal{N}$ is a neighborhood base of Y, $\mathcal{G}$ is a nonempty subset of $\mathcal{P}(X)$ such that $(\mathcal{G}, \subseteq)$ is **directed** (for any $G, H \in \mathcal{G}$, there is some $K \in \mathcal{G}$ such that $G \cup H \subseteq K$), and F is a vector subspace of $\mathrm{Fun}(X, Y)$.

(a) For any $G \in \mathcal{G}$, $N \in \mathcal{N}$, $U(G, N) := \{f \in F \mid f(G) \subseteq N\}$.

(b) If V is a vector space over $\mathbb{C}$, S is a subset of V, and $z \in \mathbb{C}$, define the set zS as

$$\{zv \in V \mid v \in S\}.$$

(c) S is **bounded** in Y if, for every neighborhood, N of 0, there is some $t \in \mathbb{C}$ such that $S \subseteq tN$.

Exercise B.26. Suppose Y is a topological vector space, X is a set, $\mathcal{N}$ is a neighborhood base of the origin of Y, $\mathcal{G}$ is a nonempty subset of $\mathcal{P}(X)$ such that $(\mathcal{G}, \subseteq)$ is directed, and F is a vector subspace of $\mathrm{Fun}(X, Y)$.

(a) There exists a unique translation-invariant topology on F such that $\{U(G, N) \mid G \in \mathcal{G} \text{ and } N \in \mathcal{N}\}$ forms a neighborhood base of the origin of F.

(b) If $\mathcal{N}_1$ and $\mathcal{N}_2$ are two neighborhood bases of the origin of Y, then $\{U(G, N) \mid G \in \mathcal{G} \text{ and } N \in \mathcal{N}_1\}$ and $\{U(G, N) \mid G \in \mathcal{G} \text{ and } N \in \mathcal{N}_2\}$ are both neighborhood bases of the same translation-invariant topology on F. Thus, this topology on F generated by such a neighborhood base is called the **topology of uniform convergence on the sets in $\mathcal{G}$**.

(c) Addition and scalar multiplication on F are continuous with respect to the topology of uniform convergence on the sets in $\mathcal{G}$ if and only if for every $G \in \mathcal{G}$, for every $f \in F$, $f(G)$ is bounded in Y.

Definition B.27. Suppose X is a set and Y is a Frechét space with sequence of seminorms p. Let $\mathrm{Bdd}(X,Y)$ be the set of all $f \in \mathrm{Fun}(X,Y)$ such that $\sup\{p(n)(f(x)) \in \mathbb{R} \mid n \in \mathbb{N} \text{ and } x \in X\} < \infty$.

Exercise B.28. If X is a set and Y is a Frechét space with sequence of seminorms p, then

(a) $\mathrm{Bdd}(X,Y)$ is a nonempty Frechét space with seminorm given by

$$p_\infty(f) = \sup\{p(n)(f(x)) \in \mathbb{R} \mid n \in \mathbb{N} \text{ and } x \in X\}$$

for all $f \in \mathrm{Bdd}(X,Y)$.

(b) If Y is a Banach space, then $\mathrm{Bdd}(X,Y)$ is a Banach space.

In either case, we say $\mathrm{Bdd}(X,Y)$ has the topology of **uniform convergence**. This is because

(c) a sequence s of $\mathrm{Bdd}(X,Y)$ converges to $f \in \mathrm{Bdd}(X,Y)$ if and only if, for every $\varepsilon > 0$, for every $K \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that, for all $n \geq M$, for every $x \in X$,

$$s(n)(x) \in B_{K,\varepsilon}(f(x)).$$

Compare Theorem B.18. We say, if a sequence s of $\mathrm{Bdd}(X,Y)$ converges to $f \in \mathrm{Bdd}(X,Y)$, s **converges uniformly** in X to f.

B.2.2 Linear Operators

Definition B.29. A **bounded map** between two topological vector spaces X and Y is any $f \in \mathcal{L}(X,Y)$ such that, if B is bounded in X, then $f(B)$ is bounded in Y. The space of all bounded maps between X and Y is denoted by $B(X,Y)$. A **bounded operator** is a bounded map between the same topological vector spaces.

Exercise B.30. If X and Y are Banach Spaces with p_Y, p_X the norm generating the topology on X, Y, respectively, then

(a) $f \in B(X,Y)$ if and only if there exists some $M \geq 0$ such that $p_Y(f(x)) \leq M p_X(x)$ for all $x \in X$.

(b) for all $f \in B(X,Y)$, $\inf\{M \geq 0 \mid p_Y(f(x)) \leq M p_X(x) \text{ for all } x \in X\}$ exists, is contained in the set, and is denoted $p_{op}(f)$.

(c) $p_{op}(f) = \sup\{p_Y(f(x)) \in \mathbb{R} \mid p_X(x) \leq 1\}$.

(d) p_{op} is a norm on $B(X,Y)$.

(e) $B(X,Y)$ is a Banach space generated by p_{op}.

(f) If $B^k(X,Y)$ is the space of all bounded k-multilinear maps from X to Y, with norm $p_{op}(f) = \sup\{p_{op}(f \circ I^i_{\mathbf{v}}) \mid \mathbf{v} \in V^n, i \in \{1.k\}\}$ using the notation of Definition A.18, then $B^k(X,Y)$ is a Banach space.

(g) $\mathrm{Bdd}(X,Y) \cap \mathcal{L}(X,Y) \subseteq B(X,Y)$. In particular, $p_{op}(f) \leq p_\infty(f)$ for all $f \in \mathrm{Bdd}(X,Y) \cap \mathcal{L}(X,Y)$.

Exercise B.31.

(a) If X and Y are Frechét spaces, then $B(X,Y) = \mathcal{L}(X,Y) \cap C(X,Y)$.

(b) If X and Y are Frechét spaces and if X is finite dimensional, then $B(X,Y) = \mathcal{L}(X,Y)$.

Exercise B.32. If $n \in \mathbb{N}$, $K \subseteq \mathbb{C}^n$ and K is closed and bounded, then K is compact. *Hint:* Prove this first for $\mathbb{R}$ using the fact that any subset of $\mathbb{R}$ is first countable, hence sequential. Consider any sequence of distinct numbers in any closed and bounded interval and inductively subdivide according to where the tail of the sequence (the image of the sequence under an infinite subset of $\mathbb{N}$) lies. Then, use the completeness of $\mathbb{R}$. Next, use the fact that $\mathbb{R}^2$ and $\mathbb{C}$ are homeomorphic. Finally, use Theorem B.21.

C Functional Analysis

Definition C.1. We suppose U is an open set of a Banach space, X, Y is a Banach space, and $f \in \mathrm{Fun}(U, Y)$.

(a) If $v \in X$, then we define the **directional derivative** of f at x with respect to the **direction** v to be the element

$$\lim_{0} \frac{f \circ (\mathrm{const}_{\mathbb{R}}^{x} + \mathrm{id}_{\mathbb{R}} \cdot \mathrm{const}_{\mathbb{R}}^{v}) - f \circ \mathrm{const}_{\mathbb{R}}^{x}}{\mathrm{id}_{\mathbb{R}}}$$

in Y, if it exists.

(b) We say f is **differentiable** at $x \in U$ if there exists some bounded map, A, such that

$$\lim_{0} \frac{p_Y \circ (f \circ (\mathrm{const}_U^x + \mathrm{id}_U) - f \circ \mathrm{const}_U^x - A)}{p_X} \to 0$$

if p_X, p_Y are the norms generating the topology on X, Y respectively. If f is differentiable at $x \in U$ for all $x \in U$, we say f is **differentiable on** U.

Exercise C.2.

(a) If a function has a directional derivative at x with respect to v, the limit is unique, denoted

$$\frac{\partial}{\partial v} f(x).$$

(b) If a function f is differentiable at a point x, the bounded map is unique, denoted

$$Df(x),$$

called the derivative of f at x.

(c) Sufficient condition for differentiability. We suppose X and Y are Banach spaces, U is open in X, $x \in U$ and $f \in \mathrm{Fun}(U, Y)$. If $\frac{\partial}{\partial v} f(x) \in Y$ for every $v \in X$, and if we define $g(v) = \frac{\partial}{\partial v} f(x)$ for every $v \in X$, then if

$$\frac{f \circ (\mathrm{const}_X^x + \mathrm{const}_X^h \cdot \mathrm{id}_X) - f \circ \mathrm{const}_X^x}{\mathrm{const}_X^h} \in \mathrm{Bdd}(X, Y)$$

for every $h \in \mathbb{R} \setminus \{0\}$, then $g \in \mathrm{Bdd}(X, Y)$. And if $g \in \mathcal{L}(X, Y)$ and

$$\frac{f \circ (\mathrm{const}_X^x + s \cdot \mathrm{id}_X) - f \circ \mathrm{const}_X^x}{s}$$

converges uniformly in X to g for every $s \in \mathrm{Fun}(\mathbb{N}, \mathbb{R} \setminus \{0\})$ which converges to 0 in $\mathbb{R}$, then f is differentiable at $x \in U$ and $Df(x)(v) = \frac{\partial}{\partial v} f(x)$ for all $v \in X$.

(d) If a function is differentiable at a point, then its directional derivatives exists at that point with respect to any direction.

(e) There exists a function whose directional derivative exists at a point with respect to every direction but the directional derivative is not linear with respect to the direction.

(f) There exists a function whose directional derivative is zero at a point with respect to every direction but the function is not continuous.

(g) There exists a function whose directional derivative is zero at a point with respect to every direction but the function is not differentiable at that point.

Definition C.3. We suppose U is an open set of a Banach space, X, Y is a Banach space, and $f \in \mathrm{Fun}(U, Y)$.

(a) We say $f \in C^1(U, Y)$ if f is differentiable on U and $Df \in C(U, B(X, Y))$.

(b) More generally, if $k \in \mathbb{N}$, we define $D^k f$ as the derivative of $D^{k-1} f$, and since $\mathcal{L}^{m+n}(X, Y)$ is isomorphic to $\mathcal{L}^m(X, \mathcal{L}^n(X, Y))$ for all $m, n \in \mathbb{N}$ by Theorem A.19, we say $f \in C^k(U, Y)$ if $D^k f \in C(U, B^k(X, Y))$. If $Y = \mathbb{R}$, we say $f \in C^k(U)$.

(c) If $C^\infty(U, Y) = \cap_{i=0}^{]} \infty C^i(U)$, the space of **smooth** functions on U to Y.

Exercise C.4. We suppose U is an open set of a Banach space, X, Y is a Banach space and $k \in \mathbb{N}$.

(a) $C^k(U, Y)$ is a vector subspace of $C^{k-1}(U, Y)$ with $C^0(U, Y) = C(U, Y)$ as the base case.

(b) $\mathcal{L}(X, Y) \subseteq C^k(U, Y)$.

(c) If $v \in X$, if the directional derivative $\frac{\partial}{\partial v}$ is defined by $\frac{\partial}{\partial v}(f)(x) = \frac{\partial}{\partial v} f(x)$ for all $f \in C^k(U, Y)$ for all $x \in U$, then $\frac{\partial}{\partial v} \in \mathcal{L}(C^k(U, Y), C^{k-1}(U, Y))$.

Theorem C.5. ADD The Inverse Function Theorem.

C.1 Distributions

C.1.1 The LF Topology

Definition C.6. Suppose X is a topological space.

(a) The **support** of a function $f : X \to \mathbb{C}$ is the set $\mathrm{Cl}(\{x \in X \mid f(x) \neq 0\})$ and is denoted by $\operatorname{supp} f$.

Definition C.7. Suppose X is a Banach space and U is open in X. And $k \in \mathbb{N} \cup \{\infty\}$.

(a) If K is a compact subset of U, denote by $C^k(K; U)$ the set of functions $f \in C^k(U)$ such that $\operatorname{supp} f \subseteq K$.

(b) Denote by $C_c^k(U)$ the subset of all $f \in C^k(U)$ such that $f \in C^k(K; U)$ for some compact $K \subseteq U$.

Exercise C.8. Suppose X is a Banach space and U is open in X. And $k \in \mathbb{N} \cup \{\infty\}$.

(a) $C_c^k(U) = \bigcup\{C^k(K; U) \mid K \text{ is compact and } K \subseteq U\}$.

Definition C.9. Suppose $n \in \mathbb{N}$ and U is open in $\mathbb{R}^n$ $U \subseteq \mathbb{R}^n$.

(a)

D Measure Theory

E Manifolds

Definition E.1. Manifolds.
Suppose (X, τ) is a topology and $n, k \in \mathbb{N}$.

(a) A **chart** on X is a pair $(U, \varphi) \in \tau \times \text{Fun}(U, E)$ with U an open set of X and φ a homeomorphism with domain U and codomain some Euclidean space E such that $\varphi(U)$ is open in E. Sometimes we refer to a chart as the homeomorphism itself rather than the pair of it with its domain.

(b) An **atlas** $\mathcal{A}$ on X is a set of charts such that, if $\mathcal{U} = P_j^{\{1\}}(\mathcal{A})$, the projection of $\mathcal{A}$ onto its first coordinate (the set of open sets from $\mathcal{A}$, $j(1) = \tau$), then $X = \bigcup \mathcal{U}$.

(c) An n-dimensional C^k **atlas**, $\mathcal{A}$, on X is an atlas, such that each member charts of the atlas are **compatible**. That is, $\mathcal{A}$ is a n-dimensional C^k **atlas** on X if $\mathcal{A}$ is an atlas on X and, for all $(U, \varphi), (V, \psi) \in \mathcal{A}$, $(U, \varphi), (V, \psi)$ are compatible in the sense the codomain of φ and ψ are n-dimensional subspaces of a Euclidean space and $\varphi \circ \psi^{-1} \in C^k(\psi(U \cap V), \varphi(U \cap V))$.

(d) An n-dimensional **maximal** C^k **atlas** on X is an n-dimensional atlas, $\mathcal{A}$, on X such that if $\mathcal{B}$ is a C^k atlas such that $\mathcal{A} \cup \mathcal{B}$ is a C^k atlas, then $\mathcal{B} \subseteq \mathcal{A}$.

(e) An n-dimensional C^k **manifold** is a separated, second countable topological space together with an n-dimensional maximal atlas on it. Sometimes we refer to the set itself as a manifold rather than as a pair with its maximal atlas when the context is clear. A zero dimensional manifold is any countable set or finite together with the discrete topology.

Definition E.2. C^k functions and the tangent space.

(a) If $(M, \mathcal{A})$ is an m-dimensional C^k manifold, and $(N, \mathcal{B})$ is an n-dimensional C^k manifold, then $f \in C^k(N, M)$ if $f \in \text{Fun}(N, M)$, is continuous, and, for all $(U, \varphi) \in \mathcal{A}$, for all $(V, \psi) \in \mathcal{B}$, $\psi \circ f \circ \varphi^{-1} \in C^k(\varphi((\psi \circ f)^{-1}(V) \cap U), \psi(V))$. We write $C^k(N, \mathbb{R}) = C^k(N)$.

(b) If M is a manifold, $p \in M$, we define the **germ** of C^k functions of M at p to be the quotient of $C^k(M)$ with the following equivalence relation: we say $f, g \in C^k(M)$ are equivalent if there exists $U \subseteq M$ open such that $p \in U$ and $f|_U = g|_U$. We denote such a germ by $C_p^k(M)$. Hence, $C_p^k(M)$ is a vector space. Moreover, there's a vector multiplication operator defined on it, by $[f][g] = [fg]$. Again, this does not depend on the choice of representatives $f, g \in C^k(M)$.

(c) If M is a manifold, $p \in M$ and $v \in (C_p^k(M))^*$, the dual of the germ of C^k functions of M at p, we say v is a **derivation** or **tangent vector** of M at p if, for all $[f], [g] \in C_p^1(M)$,

$$v([f][g]) = f(p)v([g]) + g(p)v([f]).$$

Note $f(p) = h(p)$ only if $[f] = [h]$. Hence, the right hand side of $v([f][g])$ does not depend on the choice of representatives.

(d) The tangent space of a manifold M at a point $p \in M$ is the set of all tangent vectors of M at p, denoted by T_pM.

Definition E.3. Derivatives.

(a) If $(M, \mathcal{A})$ is an m-dimensional manifold and chart $(U, \varphi) \in \mathcal{A}$, for all $p \in U$, for all $i \in \{1.m\}$ denote $[r^i \circ \phi]$ by $x^i_{(U,\varphi,p)} \in C^k_p(M)$. Usually, we write x^i instead of $x^i_{(U,\varphi,p)}$ when the dependence is clear from context. x^i, like r^i, is a **coordinate function** on M.

(b) If $(M, \mathcal{A})$ is an m-dimensional manifold and chart $(U, \varphi) \in \mathcal{A}$, for all $p \in U$, for all $i \in \{1.m\}$, for every coordinate function $x^i_{(U,\varphi,p)}$, define $\frac{\partial}{\partial x^i}\Big|_p$ by $\frac{\partial}{\partial x^i}\Big|_p ([f]) = \frac{\partial}{\partial r^i}(f \circ \varphi^{-1})(\varphi(p))$ for all $[f] \in C^k_p(M)$. $\frac{\partial}{\partial x^i}\Big|_p$ is the **partial derivative** with respect to x^i at p.

(c) If M, N are C^k manifolds and $p \in N$, the **derivative** or **differential** of $f \in C^k(N, M)$, $Df(p)$, is defined to be $Df(p)(v_p)[g] = v_p([g \circ f])$ for all $v_p \in T_pN$, for all $[g] \in C^k_{f(p)}(M)$.

Definition E.4. Submanifolds.

If $k, m \in \mathbb{N}$, $(M, \mathcal{A})$ a C^k m-dimensional manifold and $j \leq m$, a subset $S \subseteq M$ with the subspace topology is a j-dimensional C^k **regular submanifold** of M if, for all $p \in S$, there is a chart $(U, \varphi) \in \mathcal{A}$ and $I \subseteq \{1.m\}$ of size j such that $p \in U$ and

$$U \cap S = \left(P^{\{1.m\}\setminus I} \circ \varphi\right)^{-1}(\{0\}) \cap U.$$

We refer to the notation in Exercise A.9. We define the j-**adapted chart** of (U, φ) relative to S by $(U \cap S, \varphi_S)$, with

$$\varphi_S = P^I \circ \varphi\,|_{U \cap S}.$$

The φ_S is a homeomorphism on $U \cap S$ with respect to the subspace topology on S, because it's inverse is $(\varphi\,|_{U\cap S})^{-1} \circ \iota^n_I$. Moreover, if $\psi_S = P^J \circ \psi|_{V \cap S}$ is another adapted chart, then

$$\varphi_S \circ (\psi_S)^{-1} = P^I \circ \left(\varphi\,|_{U\cap S} \circ (\psi|_{V\cap S})^{-1}\right) \circ \iota_J \in C^k(\psi_S(U \cap V \cap S), \varphi_S(U \cap V \cap S))$$

by the chain rule. Also, by definition, $\varphi_S(U \cap S)$ is open in $\mathbb{R}^I_n$ since $P^I\left(\left(P^{\{1.m\}\setminus I}\right)^{-1}(\{0\})\right) = \mathbb{R}^I_n$, which implies

$$\varphi_S(U \cap S) = \mathbb{R}^I_n \cap \varphi(U).$$

Hence, a j-dimensional C^k regular submanifold S of a C^k manifold $(M, \mathcal{A})$ is a subspace topology paired with a maximal atlas of k-adapted charts of charts on $(M, \mathcal{A})$ relative to S. The number $m - j$ is called the **codimension** of S relative to M. A regular submanifold of codimension 1 is called a **hypersurface** of M.

Example E.5.

(a) If M is a manifold and $U \subseteq M$ is open, then U is a manifold of the same dimension with the subspace topology. In fact, it's a regular submanifold of M of codimension 0.

(b) $\mathbb{R}^n$ with it's standard topology and the maximal atlas containing the identity function $\mathrm{id}_{\mathbb{R}^n}$ is an n-dimensional C^k-manifold for all $k \in \mathbb{N}$. $\mathbb{R}^n$ is called flat n-space. We denote $\frac{\partial}{\partial x^i}\Big|_p$ with respect to $[r^i]_{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n}, p)}$ at p by $\frac{\partial}{\partial r^i}\Big|_p$.

Theorem E.6. (C^1 remainder theorem) If M is an m-dimensional manifold, $p \in M$, (U, φ) a member of the maximal atlas of M such that $p \in U$ and $\varphi(U)$ is an open disk in $\mathbb{R}^m$ centered at $\varphi(p)$, and $[f] \in C^1_p(M)$, then

$$f|_U = \text{const}_U^{f(p)} + \sum_{i=1}^{m} (r^i \circ \varphi - \text{const}_U^{r^i \circ \varphi(p)}) \cdot g_i$$

for some $g_i \in C^k(U)$ such that $g_i(p) = \frac{\partial}{\partial x^i}\Big|_p ([f])$ for all $i \in \{1.m\}$. Here, $f|_U$ is the restriction of f to U. r^i is the ith coordinate function on $\mathbb{R}^m$, $\text{const}_U^{f(p)}$ is the constant function $f(p)$ on U, and so on. The right hand side is function addition $+$, $\sum$, function subtraction, $-$, and function multiplication $\cdot$.

Proof. From [Tu, '11, pg 6].

We consider $h = f \circ \varphi^{-1}$. Suppose the radius of $\varphi(U)$ is $s \in \mathbb{R}$. By the Chain Rule,

$$\frac{\partial}{\partial t} h(\varphi(p) + \frac{t}{s}(r - \varphi(p))) = \frac{1}{s} \sum_{i=1}^{m} \frac{\partial}{\partial r^i}(h)(\varphi(p) + t(r - \varphi(p)))(r^i(r) - r^i \circ \varphi(p))$$

for all $t \in [0, s]$, for all $r \in \varphi(U)$.

Define, for all $i \in \{1.m\}$,

$$g_i \circ \varphi^{-1}(r) = \frac{1}{s} \int_0^s \frac{\partial}{\partial r^i} h\left(\varphi(p) + \frac{t}{s}(r - \varphi(p))\right) dt \text{ for all } r \in \varphi(U).$$

Then, for all $i \in \{1.m\}$, $g_i \in C^k(U)$ by the fundamental theorem of calculus since $h \in C^k(\varphi(U))$,

$$g_i(p) = \frac{\partial}{\partial x^i}\Big|_p ([f])$$

and

$$h(r) - f(p) = \sum_{i=1}^{m} (r^i(r) - r^i \circ \varphi(p)) g_i \circ \varphi^{-1}(r)$$

for all $r \in \varphi(U)$. □

Exercise E.7. Suppose N, M are n, m-dimensional C^k manifolds, respectively, $p \in N$ and $f \in C^k(N, M)$.

(a) $T_p N$ is a vector subspace of $(C^k_p(N))^*$.

(b) $\frac{\partial}{\partial x^i}\Big|_p \in T_p N \subset \left(C^k_p(N)\right)^* \subset \text{Fun}\left(C^1_p(N), \mathbb{R}\right)$ for all $i \in \{1.n\}$.

(c) $\left\{ \frac{\partial}{\partial x^1}\Big|_p \cdot \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of $T_p N$. *Hint:* Apply the C^1 remainder theorem.

(d) $Df(p) \in \mathcal{L}\left(T_p N, T_{f(p)} M\right)$.

Definition E.8. If M, N are C^k manifolds, $f \in C^k(N, M)$, $p \in N$ and $c \in M$, then

(a) p is a **regular point** of f if $Df(p)$ is surjective.

(b) c is a **regular value** if either $f^{-1}(c)$ is empty or $f^{-1}(c)$ consists entirely of regular points of f.

(c) f is a k-**diffeomorphism** if f is invertible and $f^{-1} \in C^k(M, N)$.

Exercise E.9. Suppose $n, m, k \in \mathbb{N}$, N, M are n, m-dimensional manifolds, respectively, $m \leq n$, $c \in M$, and $f \in C^k(N, M)$.

(a) If c is a regular value of f and is a member of its range, then $f^{-1}(c) \subseteq N$ is an $(n-m)$-dimensional regular submanifold of N. *Hint: The inverse function theorem applies.*

(b) If $f : N \to f(N)$ is a homeomorphism with $f(N)$ the subspace topology with respect to M, and $Df(p)$ is injective for every $p \in N$, then $f(N)$ is an n-dimensional regular submanifold of M.

(c) The function mapping real x to x^3 is a bijection from $\mathbb{R}$ to $\mathbb{R}$ but not a 1-diffeomorphism.

(d) If $S \subseteq N$, is a submanifold, and f is a diffeomorphism, then $f|_S \in C^k(S, f(S))$ is a diffeomorphism.

Theorem E.10. If S is a submanifold of M, $(U \cap S, \varphi_S)$ is an j-adapted chart relative to S, $p \in U \cap S$, $\eta = \varphi_S$, and $V = \eta(U \cap S)$, then

$$D(\iota_{S,M})(p)(T_pS) = D\eta^{-1}(\eta(p))(\mathbb{R}^j).$$

Proof.

$$D(\eta \circ \iota_{S,M})(p) \left(\left. \frac{\partial}{\partial x^i} \right|_p \right)([g]) = \left. \frac{\partial}{\partial x^i} \right|_p ([g \circ \eta \circ \iota_{S,M}]) = \frac{\partial}{\partial r^i}(g \circ \eta \circ \iota_{S,M} \circ \varphi^{-1})(\varphi(p)).$$

Any real-valued function f on V can be written as $g \circ \eta \circ \iota_{S,M} \circ \varphi^{-1}$ for some real-valued function g on $U \cap S$. □

Definition E.11. Suppose $n, k \in \mathbb{N}$.

(a) If N,M are manifolds and S is a subset of N, we say $f \in C^k(S, M)$ if $f \in \mathrm{Fun}(S, M)$ and there exists open $U \subseteq N$ and $\tilde{f} \in C^k(U, M)$ such that $S \subseteq U$ and $f = \tilde{f}|_S$.

(b) If $n \in \mathbb{N}$, the **upper-half space** $\mathbb{R}^n_+$ is the set $\{x \in \mathbb{R}^n \mid x(n) > 0\}$. Thus, its boundary is $\mathrm{bdy}\, \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x(n) = 0\}$ and its closure is $\mathrm{Cl}\left(\mathbb{R}^n_+\right) = \{x \in \mathbb{R}^n \mid x(n) \geq 0\}$.

(c) A n-dimensional C^k manifold with boundary is a separated, second countable topological space, together with a maximal C^k atlas of compatible charts whose images are open in $\mathrm{Cl}\left(\mathbb{R}^n_+\right)$, and $\mathrm{Cl}\left(\mathbb{R}^n_+\right)$ is considered to employ the subspace topology of $\mathbb{R}^n$.

(d) If $(M, \mathcal{A})$ is an n-dimensional C^k manifold with boundary, then define the **manifold interior**, denoted $\mathrm{Int}_{\mathrm{Mfld}}(M)$, to be the set of all points $p \in M$ such that there is a chart $(U, \varphi) \in \mathcal{A}$ such that $p \in U$ and $\varphi(p) \in \mathbb{R}^n_+$. The **manifold boundary** of M is $\mathrm{bdy}_{\mathrm{Mfld}}(M)$, defined as the set of all points $p \in M$ such that there is a chart $(U, \varphi) \in \mathcal{A}$ such that $p \in U$ and $\varphi(p) \in \mathrm{bdy}\, \mathbb{R}^n_+$.

Exercise E.12.

(a) From [Tu, '11, Theorem 22.3] If U is open in $\mathbb{R}^n$, S is a subset of $\mathbb{R}^n$, and $f : U \to S$ is a diffeomorphism, then S is open in $\mathbb{R}^n$. Hint: The inverse function theorem applies.

(b) From [Tu, '11, Proposition 22.4] If U, V are open in $\mathrm{Cl}\left(\mathbb{R}^n_+\right)$ and $f : U \to V$ is a diffeomorphism, then $f(U \cap \mathbb{R}^n_+) = V \cap \mathbb{R}^n_+$ and $f(U \cap \mathrm{bdy}\, \mathbb{R}^n_+) = V \cap \mathrm{bdy}\, \mathbb{R}^n_+$.

Theorem E.13. If $m, k \in \mathbb{N}$ and M is an m-dimensional C^k manifold with boundary, then

(a) the manifold interior of M, $\mathrm{Int}_{\mathrm{Mfld}}(M)$, is an m-dimensional C^k manifold.

(b) if the manifold boundary of M, $\mathrm{bdy}_{\mathrm{Mfld}}(M)$, is not empty, then it is an hypersurface of M.

Proof.

(a) We know $\mathrm{Int}_{\mathrm{Mfld}}(M)$ is nonempty if M is. This is because if (U, φ) is any chart of M, then $\varphi(U)$, since φ is a homeomorphism, is open in $\mathrm{Cl}\left(\mathbb{R}^m_+\right)$. Hence, $\varphi(U) = W \cap \mathrm{Cl}\left(\mathbb{R}^m_+\right)$ for some open W in $\mathbb{R}^m$. If $x \in W \cap \mathrm{bdy}\, \mathbb{R}^m_+$, then there is some disk $D \subseteq W$ such that $x \in D$ and $D \cap \mathbb{R}^m_+ \neq \emptyset$. Hence, $\varphi(U) \cap \mathbb{R}^m_+$ is not empty. We can then consider $\varphi|_{\varphi^{-1}(\mathbb{R}^m_+)}$ as a chart of $\mathrm{Int}_{\mathrm{Mfld}}(M)$.

(b) If $p \in \mathrm{bdy}_{\mathrm{Mfld}}(M)$, then there is a chart (U, φ) in the atlas of M such that $\varphi(p) \in \mathrm{bdy}\, \mathbb{R}^m_+$. Now $\mathrm{bdy}\, \mathbb{R}^m_+ = \left(P^{\{1.m\}\setminus\{1.m-1\}}\right)^{-1}(\{0\})$ using the notation of Exercise A.9. Hence,

$$U \cap \mathrm{bdy}_{\mathrm{Mfld}}(M) = U \cap \left(P^{\{1.m\}\setminus\{1.m-1\}} \circ \varphi\right)^{-1}(\{0\})$$

by the above exercise.

□

Definition E.14. Suppose $n, k \in \mathbb{N}$. If $\Omega \subseteq \mathbb{R}^n$ is bounded, open, and connected, then Ω is a C^k-domain if, for every $x \in \mathrm{bdy}\, \Omega$, there is a disk, D, centered at x, and a k-diffeomorphism ψ with domain D such that

$$\psi(D \cap \Omega) \subset \mathbb{R}^n_+$$

and

$$\psi(D \cap \mathrm{bdy}\, \Omega) \subset \mathrm{bdy}\, \mathbb{R}^n_+ .$$

Lemma E.15. If $k \in \mathbb{N}$, M, N are manifolds, $S \subseteq N$, and $f \in C^k(S, M)$, then, for any $T \subseteq S$, the restriction $f|_T \in C^k(T, M)$.

Proof. Since $f \in C^k(S, M)$, there exists open $U \subseteq N$ such that $S \subseteq U$ and $\tilde{f} \in C^k(U, M)$ such that $f = \tilde{f}|_S$. Hence, $f|_T = \tilde{f}|_S|_T = \tilde{f}|_T$. □

Theorem E.16. Suppose $n, k \in \mathbb{N}$, $n > 1$, and Ω is a C^k-domain in $\mathbb{R}^n$.

(a) $\mathrm{bdy}\, \Omega$ is a hypersurface of $\mathbb{R}^n$.

(b) With the subspace topology on $\mathbb{R}^n$, $\mathrm{Cl}(\Omega)$ is a C^k manifold with boundary, whose manifold interior is Ω and whose manifold boundary is $\mathrm{bdy}\, \Omega$.

Proof.

(a) Notice $\mathrm{bdy}\, \mathbb{R}^n_+ = \left(P^{\{1.n\}\setminus\{1.n-1\}}\right)^{-1}(\{0\})$. If $x \in \mathrm{bdy}\, \Omega$ find disk D center x and diffeomorphism ψ with domain D such that

$$\psi(D \cap \mathrm{bdy}\, \Omega) \subset \left(P^{\{1.n\}\setminus\{1.n-1\}}\right)^{-1}(\{0\}).$$

Such D and ψ exist because Ω is a C^k-domain. Since ψ is a diffeomorphism, (D, ψ) is in the maximal atlas of n flat space. And

$$D \cap \mathrm{bdy}\, \Omega \subset \left(P^{\{1.n\}\setminus\{1.n-1\}} \circ \psi\right)^{-1}(\{0\}).$$

Moreover,

$$D \cap \operatorname{bdy}\Omega = D \cap \left(P^{\{1.n\}\setminus\{1.n-1\}} \circ \psi\right)^{-1}(\{0\})$$

as we'll prove in the lemma below.

(b) Define $m = \min\{a \in \mathbb{R} \mid a = x(n) \text{ for some } x \in \operatorname{Cl}(\Omega)\}$. Such an m exists since $\operatorname{Cl}(\Omega)$ is closed and bounded, hence compact, and the function r^n is continuous. We consider the map $\varphi = \mathrm{id}_\Omega + \mathrm{const}_\Omega^{(|m|+1)\mathbf{e}_n}$. Then φ is a diffeomorphism of Ω onto an open subset of $\mathbb{R}^n_+$. For every $x \in \operatorname{bdy}\Omega$, there exists a disk D and an diffeomorphism ψ such that

$$\psi(D \cap \operatorname{Cl}(\Omega)) \subseteq \operatorname{Cl}\left(\mathbb{R}^n_+\right).$$

Lemma E.17.

$$\psi(D \cap \operatorname{Cl}(\Omega)) = \psi(D) \cap \operatorname{Cl}\left(\mathbb{R}^n_+\right).$$

Then, since ψ is a homeomorphism and D is open in $\mathbb{R}^n$, $\psi(D \cap \operatorname{Cl}(\Omega))$ is open in $\operatorname{Cl}\left(\mathbb{R}^n_+\right)$ with the subspace topology.

Proof. Suppose $\mathbb{R}^n_+ \cap \psi(D) \cap \psi(D \setminus \operatorname{Cl}(\Omega)) \neq \emptyset$. Then, if $\mathbb{R}^n_+ \cap \psi(D)$ is connected, by Exercise B.11, $\mathbb{R}^n_+ \cap \psi(D) \cap \operatorname{bdy}\psi(D \setminus \operatorname{Cl}(\Omega)) \neq \emptyset$ follows. If $\mathbb{R}^n_+ \cap \psi(D)$ is not connected, then there is some connected component $U \subseteq \mathbb{R}^n \cap \psi(D)$ such that $U \cap \psi(D \setminus \operatorname{Cl}(\Omega)) \neq \emptyset$. Hence, by Exercise B.11 $U \cap \operatorname{bdy}\psi(D \setminus \operatorname{Cl}(\Omega)) \neq \emptyset$.

But this is a contradiction, since we claim $\psi(D) \cap \operatorname{bdy}\psi(D \setminus \operatorname{Cl}(\Omega)) \subseteq \operatorname{bdy}\mathbb{R}^n_+$. $y \in \psi(D) \cap \operatorname{bdy}\psi(D \setminus \operatorname{Cl}(\Omega))$ if and only if, for all open $W \subseteq \psi(D)$, if $y \in W$, then

$$W \cap \psi(D \setminus \operatorname{Cl}(\Omega)) \neq \emptyset$$

and

$$W \setminus \psi(D \setminus \operatorname{Cl}(\Omega)) \neq \emptyset$$

by Exercise B.8, if and only if, for all open $V \subseteq D$, if $\psi^{-1}(y) \in V$, then

$$V \cap \operatorname{Cl}(\Omega) \neq \emptyset$$

and

$$V \setminus \operatorname{Cl}(\Omega) \neq \emptyset$$

if and only if $\psi^{-1}(y) \in D \cap \operatorname{bdy}\operatorname{Cl}(\Omega) = D \cap \operatorname{bdy}\Omega$. Hence, $y \in \operatorname{bdy}\mathbb{R}^n_+$. Hence,

$$\psi(D) \cap \operatorname{bdy}\psi(D \setminus \operatorname{Cl}(\Omega)) \subseteq \operatorname{bdy}\mathbb{R}^n_+.$$

And subsequently

$$U \cap \operatorname{bdy}\psi(D \setminus \operatorname{Cl}(\Omega)) = \emptyset$$

and

$$\mathbb{R}^n_+ \cap \psi(D \setminus \operatorname{Cl}(\Omega)) = \emptyset.$$

Hence,

$$\psi(D \cap \operatorname{Cl}(\Omega)) = \psi(D) \cap \operatorname{Cl}\left(\mathbb{R}^n_+\right).$$

□

Consider $\eta = \psi|_{D\cap \mathrm{Cl}(\Omega)}$. Then η is a homeomorphism of $D\cap \mathrm{Cl}(\Omega)$ with the subspace topology of $\mathrm{Cl}(\Omega)$ onto $\psi(D \cap \mathrm{Cl}(\Omega))$ with the subspace topology on $\mathrm{Cl}\left(\mathbb{R}^n_+\right)$. Now, $\psi(D \cap \mathrm{Cl}(\Omega))$ is open in $\mathrm{Cl}\left(\mathbb{R}^n_+\right)$ because $\psi(D\cap \mathrm{Cl}(\Omega)) = \psi(D)\cap \mathrm{Cl}\left(\mathbb{R}^n_+\right)$ and $\psi(D)$ is open in $\mathbb{R}^n$. Hence, η is a chart on $\mathrm{Cl}(\Omega)$. If η_1 and η_2 are two such charts on $\mathrm{Cl}(\Omega)$, such that $\eta_i = \psi_i|_{D_i\cap \mathrm{Cl}(\Omega)}$, for diffeomorphisms $\psi_i \in C^k(D_i, \psi_i(D_i))$ if $i \in \{1, 2\}$, then

$$\eta_1\circ\eta_2^{-1} \in C^k(\eta_2(D_1 \cap D_2 \cap \mathrm{Cl}(\Omega)), \eta_1(D_1 \cap D_2 \cap \mathrm{Cl}(\Omega)))$$

because

$$\eta_1\circ\eta_2^{-1} = \psi_1|_{D_1\cap \mathrm{Cl}(\Omega)}\circ \left(\psi_2|_{D_2\cap \mathrm{Cl}(\Omega)}\right)^{-1} = \psi_1|_{D_1\cap \mathrm{Cl}(\Omega)}\circ\psi_2^{-1}|_{\eta_2(D_2\cap \mathrm{Cl}(\Omega))} = (\psi_1\circ\psi_2^{-1})|_{\eta_2(D_1\cap D_2\cap \mathrm{Cl}(\Omega))}.$$

And the identity plus a constant composed with any diffeomorphism is again a diffeomorphism. Hence, this described collection of charts forms an atlas, part of a maximal atlas we equip on $\mathrm{Cl}(\Omega)$. And with this atlas, the manifold boundary of $\mathrm{Cl}(\Omega)$ is its topological boundary bdy Ω and its manifold interior is its topological interior Ω.

□

F Ordinary Differential Equations

Theorem F.1. If U is an open subset of $\mathbb{R}^m$, V is an open subset of $\mathbb{R}^n$, $g \in C^1(U, V) \cap B(U, V)$, $t_0 \in \mathbb{R}$, and $F \in C^1(V\times\mathbb{R}, \mathbb{R}^n)$, then there exists an open set J, containing t_0, contained in $\mathbb{R}$, and a unique solution $y \in C^1(U \times J, \mathbb{R}^n)$ of

$$\begin{cases} \frac{\partial}{\partial r^{m+1}} y = F\circ(y, r^{m+1}) & \text{on} \quad U \times J \\ y = g\circ P^{\{1.m\}} & \text{on} \quad U \times \{t_0\}. \end{cases} \tag{F.2}$$

Here, r^{m+1} is the projection of $\mathbb{R}^{m+1}$ onto $\mathbb{R}$, $P^{\{1.m\}}$ is the projection of $\mathbb{R}^{m+1}$ onto $\mathbb{R}^m$ and (y, r^{m+1}) is a rectangular product of functions whose codomain is considered a subset of $V\times\mathbb{R} \subseteq \mathbb{R}^{n+1}$. Specifically,$(y, r^{m+1}) = (g_1, \cdots, g_{m+1})$ if $g_i = r^i\circ y$ for $i \in \{1.n\}$ and $g_{n+1} = r^{m+1}$.

[Taylor, '11, Theorems 2.1 - 2.2]

Proof. Fix $K > 0$ such that

$$\{y \in \mathbb{R}^n \mid |\mathrm{const}^y_U - g|_{C(U,\mathbb{R}^n)} \leq K\} \subseteq V.$$

And fix $M > 0$ and $I \subseteq \mathbb{R}$ center t_0 such that

$$\sup_{V\times I} |F| \leq M.$$

And fix $L > 0$ such that $|F(y_1, t) - F(y_2, t)| \leq L|y_1 - y_2|$ for all $t \in I$, for all $y_1, y_2 \in V$. Such an L exists since $F \in C^1(V \times I, \mathbb{R}^n)$. Hence, if we fix $\varepsilon \leq \min\left\{\frac{K}{M}, \text{radius of } I, \frac{1}{L}\right\}$, and define $J = B_\varepsilon(t_0) \subseteq \mathbb{R}$,

$$X = \left\{u \in C(U \times J, \mathbb{R}^n) \mid u|_{U\times\{t_0\}} = g\circ P^{\{1.m\}} \text{ and } |u - g\circ P^{\{1.m\}}|_{C(U\times J,\mathbb{R}^n)} \leq K\right\},$$

and

$$T(u)(x, t) = g(x) + \int_{t_0}^{t} F(u(x, s), s)ds,$$

for all $u \in X$, $(x,t) \in U \times J$, then X is a complete metric space because it's a closed set of the Banach space $C(U \times J, \mathbb{R}^n) \cap B(U \times J, \mathbb{R}^n)$, range$(T) \subseteq X$ and T is a contraction. Hence, by the contraction mapping principle, there is a unique fixed point $y \in X$ such that $T(y) = y$. Hence,

$$y$$

is the unique solution of F.1.

$\square$

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