Curves and Series Notes

Subject to change.

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1 Sequences and Series

1.1 Sequences

Definition 1.1. Recall a **sequence** is a function whose domain is an infinite subset of the natural numbers. Sequences and counting are highly correlated, which make them seemingly simple objects. The nth **term** of a sequence is the output of the sequence whose input is n. A **recursive sequence** is a sequence whose initial term is explicit and whose nth term is defined in terms of its earlier terms. If a sequence's domain is $\{n_0, n_0+1, n_0+2, \dots\}$, and its nth term is denoted by a_n, then we sometimes denote it as $\{a_n\}_{n=n_0}^{\infty}$.

Example 1.2. The nth term of the **factorial sequence** is denoted by $n!$ and is read 'n factorial'. This sequence $\{n!\}_{n=0}^{\infty}$ is defined recursively as

$$0! = 1$$

and

$$n! = n \cdot (n-1)!$$

for any $n > 1$. The fourth term of the factorial sequence is 24 because $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. 52!, the number of ways a deck of 52 cards can be shuffled, is an incredibly large number. It's so large that, any time you shuffle a deck of cards, the result is most likely an arrangement that has never occurred before.

Example 1.3. A **geometric sequence** is the restriction of an exponential function to the natural numbers. By our definition, an exponential function is of the form $f(x) = a \cdot b^x$ if $-\infty < x < \infty$ for some real number a (the initial term) and some real, positive, nonone number b (the base).

Exercise 1.4. Describe any geometric sequence recursively.

Example 1.5. We can sometimes approximate an irrational number with a recursive sequence of rational numbers. For example, if

$$a_0 = 2$$

and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

for any $n \geq 0$, then

$$a_1 = \frac{3}{2},$$

$$a_2 = \frac{17}{12},$$

and

$$a_3 = \frac{577}{408} \approx 1.414215686 \approx \sqrt{2}.$$

In fact, the terms a_n can be made as close to $\sqrt{2}$ as we wish given we make n sufficiently large.

Definition 1.6. (a) If L is any real number such that the terms a_n can be made as close to L as we wish, given we make n sufficiently large, then we say the **limit** of the sequence $\{a_n\}_{n=n_0}^{\infty}$ **exists** and is equal to L. Formally, if it exists, the limit of $\{a_n\}_{n=n_0}^{\infty}$ is any number L such that, for any $\varepsilon > 0$, there is some natural number N such that, if $n \geq N$ and n is in the domain of a_n, then $|a_n - L| < \varepsilon$. We use the notation

$$\lim_{n\to\infty} a_n = L$$

for such a situation and we say the sequence $\{a_n\}_{n=n_0}^{\infty}$ **converges** to L.

(b) If a_n can be made as large and positive as we wish, given we make n sufficiently large, we say the sequence $\{a_n\}_{n=n_0}^{\infty}$ **diverges to** ∞. Formally, we say $\{a_n\}_{n=n_0}^{\infty}$ **diverges to** ∞ if, for any $M > 0$, there is some natural number N such that, if $n \geq N$, then $a_n > M$.

Example 1.7. The following are examples from precalculus.

(a) If $a_n = 1$ for all natural number n, then

$$\lim_{n\to\infty} a_n = 1.$$

(b)

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

To demonstrate formally: For any $\varepsilon > 0$, find any $N > \frac{1}{\varepsilon}$. Then if $n \geq N$, then $|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

(c)

$$\lim_{n\to\infty} 2^{-n} = 0.$$

To demonstrate formally: For any $\varepsilon > 0$, find any $N > -\ln(\varepsilon)$. Then if $n \geq N$, then $|2^{-n} - 0| = 2^{-n} \leq 2^{-N} < 2^{\ln(\varepsilon)} = \varepsilon$. Here we twice used the fact that an exponential function is an increasing function if its base is greater than one.

(d)

$$\lim_{n\to\infty} \frac{2n+1}{n-1} = 2.$$

To demonstrate formally: For any $\varepsilon > 0$, find any $N > \frac{2}{\varepsilon} + 1$. Then if $n \geq N$, then $|\frac{2n+1}{n-1} - 2| = \frac{2}{n-1} \leq \frac{2}{N-1} < \varepsilon$.

1.1.1 Limit laws

The following are ways to obtain new limits from old ones. The first two items establish the **linearity** of the limit operator.

Theorem 1.8. (Limit laws) Suppose the limits $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist and suppose c is a real number. Then

(a)

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n.$$

(b)

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(c)

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

Proof. Suppose the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and suppose c is a real number. Denote $\lim_{n\to\infty} a_n$ by a and $\lim_{n\to\infty} b_n$ by b. Let $\varepsilon > 0$ be arbitrary.

(a) Since a exists, there exists N such that $|a_n - a| < \frac{\varepsilon}{|c|+1}$ for all $n \geq N$. Hence,

$$|c \cdot a_n - c \cdot a| \leq |c| \cdot |a_n - a| < \frac{|c|}{|c|+1} \cdot \varepsilon < \varepsilon\,.$$

The nonstrict inequality follows from the Cauchy-Schwarz inequality: $|ab| \leq |a||b|$ for all real numbers a and b.

(b) Since a and b exist, there exists N_1 such that, for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2}$ and N_2 such that, for all $n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2}$. Then if we define $N = \max\{N_1, N_2\}$, then

$$|a_n + b_n - (a+b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \varepsilon$$

if $n \geq N$. The nonstrict inequality follows from the **triangle inequality**: $|x+y| \leq |x| + |y|$ for all real numbers x and y.

(c) Since a exists, there exists N_1 such that, for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2(|b|+1)}$. Hence, $|a_n| \leq \frac{\varepsilon}{2(|b|+1)} + |a|$ for all $n \geq N_1$. This step follows from the **reverse triangle inequality**: $||x| - |y|| \leq |x+y|$ for all real numbers x and y, which can be proved from the triangle inequality.

Since b exists, there exists N_2 such that $|b_n - b| < \frac{\varepsilon}{2\cdot\left(\frac{\varepsilon}{2(|b|+1)} + |a| + 1\right)}$ for all $n \geq N_2$.

Then for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} |a_n \cdot b_n - a \cdot b| &= |a_n \cdot b_n - a_n \cdot b + a_n \cdot b - a \cdot b| \\ &\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| \\ &< \left(\frac{\varepsilon}{2(|b|+1)} + |a|\right) \cdot \frac{\varepsilon}{2\cdot\left(\frac{\varepsilon}{2(|b|+1)} + |a| + 1\right)} + |b| \cdot \frac{\varepsilon}{2(|b|+1)} \\ &< \varepsilon\,. \end{aligned}$$

The nonstrict inequality is again from the triangle inequality.

□

Exercise 1.9. Using the ideas from the previous theorem's proof, explain, if $\lim_{n\to\infty} b_n \neq 0$, then $\lim_{n\to\infty} \frac{1}{b_n}$ exists and equals $\frac{1}{\lim_{n\to\infty} b_n}$.

Exercise 1.10. Using the previous theorem and exercise, explain, if the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and if $\lim_{n\to\infty} b_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Definition 1.11. If I is an interval of $\mathbb{R}$, a function $f : I \to \mathbb{R}$ is **continuous** if, for any sequence $\{a_n\}_{n=n_0}^{\infty}$, if $a_n \in I$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n \in I$, then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$.

Exercise 1.12. Justify the following statements.

(a) If $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n$ is nonnegative.

(b) If $a_n < b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

(c) (Sandwich Theorem) If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Example 1.13. We recall the recursive sequence $\{a_n\}_{n=0}^{\infty}$ in Example 1.5: $a_0 = 2$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for any $n \in \mathbb{N}$. This is a sequence of rational numbers approaching the irrational number $\sqrt{2}$. If the limit of a_n exists, then the limit of a_{n+1} exists. In particular, if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} a_{n+1} = a$. Hence, $a = a/2 + 1/a$. This implies $a^2 = 2$. Hence, $a = \sqrt{2}$. Of course, we need to show the limit exists (outlined in Exercise 1.33) and that $a_n > 0$ for all n to conclude $a \neq -\sqrt{2}$ using the previous exercise.

1.1.2 Nonexamples

Example 1.14.

(a) $a_n = n^2 + 1$. This sequence does not have a real-valued limit, but it does approach $+\infty$.

(b) $a_n = (-1)^n$. This sequence does not have a limit. It alternates between -1 and 1 infinitely often. There is no number for which a_n is arbitrarily close to given a large enough n.

Exercise 1.15. The following exercise is adapted from one of Torsten Erhrardt's. For each of the following sequences, does their limit exist? If so, what is it? If not, why not?

(a)

$$a_n = \sin\left(\frac{2\pi n^2}{3(n+1)}\right)$$

(b)

$$b_n = a_{3n}$$

(c)

$$c_n = a_{3n+1}$$

(d)

$$d_n = a_{3n+2}$$

Exercise 1.16. The **Logistic map** with parameter r is the recursive sequence, for each $x_0 \in [0,1]$: $x_{n+1} = rx_n(1 - x_n)$.

1. Demonstrate $\lim_{n\to\infty} x_n = 0$ if $r \in [0,1]$ and $x_0 \in [0,1]$.
2. Demonstrate $\lim_{n\to\infty} x_n = \frac{r-1}{r}$ if $r \in [1,3]$ and $x_0 \in (0,1)$.
3. Verify $x_n = \sin^2(2^n\theta)$ with $\theta = \sin^{-1}(\sqrt{x_0})$ is the solution to the logistic map with parameter 4.
4. Write some code to gain some evidence in favor of the sensitivity of initial conditions of the logistic map when its parameter is roughly > 3.57.

1.1.3 Monotonicity

Definition 1.17.

(a) A sequence $\{a_n\}_{n=0}^{\infty}$ is **monotone increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

(b) A sequence $\{a_n\}_{n=0}^{\infty}$ is **monotone decreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.

(c) A sequence $\{a_n\}_{n=0}^{\infty}$ is **bounded** if there is some $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 1.18. Any bounded monotone increasing or monotone decreasing sequence converges.

Proof. If $\{a_n\}_{n=0}^{\infty}$ is monotone increasing, then the limit of this sequence is the least upper bound of the set $\{a_n \mid n \in \mathbb{N}\}$, which exists by the boundedness of this set along with Theorem B.6. If $\{a_n\}_{n=0}^{\infty}$ is monotone decreasing, then the limit of this sequence is the least upper bound of the set $\{-a_n \mid n \in \mathbb{N}\}$. □

Hence, if $\{a_n\}_{n=0}^{\infty}$ is monotone increasing and converges, then $a_n \leq \lim_{n\to\infty} a_n$ for all $n \in \mathbb{N}$.

Exercise 1.19. Justify the following statement. If a monotone increasing sequence is not bounded, it diverges to ∞.

1.1.4 Subsequences

Definition 1.20. A **subsequence** of a sequence $\{a_n\}_{n=0}^{\infty}$ is any sequence $\{b_n\}_{n=0}^{\infty}$ such that there exists a function $s : \mathbb{N} \to \mathbb{N}$ such that $s(n) < s(n+1)$ for all n and $b_n = a_{s(n)}$.

Theorem 1.21. If a sequence converges, then any subsequence of it converges to the same limit.

Proof. If $\{a_n\}_{n=0}^{\infty}$ converges to a, then for any $\varepsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N_1$.

If $\{b_n\}_{n=0}^{\infty}$ is a subsequence of $\{a_n\}_{n=0}^{\infty}$ such that $b_n = a_{s(n)}$ and $s(n) < s(n+1)$ for all n, then if $N = s^{-1}(N_1)$, then if $n \geq N$, then $s(n) \geq N_1$ by the increasingness of s. Hence, $|b_n - a| = |a_{s(n)} - a| < \varepsilon$. Hence, $\{b_n\}_{n=0}^{\infty}$ converges to a. □

1.2 Series

Definition 1.22. Given a sequence $\{a_n\}_{n=n_0}^{\infty}$, a **partial sum** S_n with respect to a_n is defined recursively as

$$S_1 = a_1$$

and

$$S_{n+1} = a_{n+1} + S_n.$$

Then

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

We introduce **summation notation**. We write

$$S_n = \sum_{i=1}^{n} a_i.$$

In other words,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

Example 1.23.

(a) If $a_n = 1 + n$, then $S_3 = \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = 1 + 1 + (1 + 2) + (1 + 3) = 2 + 3 + 4 = 9$.

(b)

$$\sum_{i=1}^{5} 3i^2 = 3(1) + 3(4) + 3(9) + 3(16) + 3(25) = 165.$$

(c)

$$\sum_{i=0}^{3} (-1)^i (i-1) = (-1)^0(0-1) + (-1)^1(1-1) + (-1)^2(2-1) + (-1)^3(3-1) = -1-0+1-2 = -2.$$

(d)

$$\sum_{i=0}^{6} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.71805555556.$$

This is close to the number e.

Exercise 1.24. Write each of the following as a rational number.

(a) $\sum_{i=1}^{12} 5$.

(b) $\sum_{i=0}^{2} \frac{1}{2+i}$.

(c) $\sum_{k=2}^{5} \frac{1}{2+k}$.

(d) $\sum_{j=-1}^{5} \frac{1}{2+j}$.

(e) $\sum_{i=1}^{n}(a_i + b_i)$ if $\sum_{i=1}^{n} a_i = 12$ and $\sum_{i=1}^{n} b_i = -15$.

Example 1.25. The i, j, k in the previous exercise is called the **index** of the sum. Indices are interchangeable if that is all we change. For example,

$$\sum_{i=0}^{3} \sin(i) = \sin(0) + \sin(1) + \sin(2) + \sin(3) = \sum_{k=0}^{3} \sin(k).$$

Definition 1.26. A **series** is a sequence of partial sums. If $\{a_n\}_{n=n_0}^{\infty}$ is a sequence, then the **series formed by** $\{a_n\}_{n=n_0}^{\infty}$ is the sequence $\{\sum_{k=n_0}^{n} a_k\}_{n=n_0}^{\infty}$.

1.2.1 Geometric Series

Exercise 1.27. $\lim_{n\to\infty} b^n = 0$ if and only if $|b| < 1$. Prove this statement by following these steps.

(a) Suppose $|b| \geq 1$ and there is some $M > 0$ for which $|b|^n \leq M$ for all $n \in \mathbb{N}$.

(b) Then the least upper bound of the the set $\{|b|^n \mid n \in \mathbb{N}\}$ exists (justify). Label this least upper bound a.

(c) Then, for all $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that $a - |b|^n < \varepsilon$ (justify).

(d) Justify $|b|^{n+1} - |b|^n < \varepsilon$ for all $n \in \mathbb{N}$.

(e) Conclude, for all $\varepsilon > 0$, $0 \leq |b| - 1 < \varepsilon$. This implies $|b| = 1$.

(f) Conclude, if $|b| > 1$, then, for all $M > 0$, there exists some $n \in \mathbb{N}$ such that $|b|^n > M$. Hence, if $|b| > 1$, $\lim_{n\to\infty} |b|^n = \infty$.

(g) Conclude, if $|b| < 1$, then $\lim_{n\to\infty} b^n = 0$.

Example 1.28. A geometric series is a sequence of partial sums formed by a geometric sequence. If $a_n = c \cdot b^n$ for $n = 0, 1, 2, 3, \ldots$, $c \in \mathbb{R}$, $b > 0$ and $b \neq 1$, then

$$S_n = \sum_{i=0}^{n} c \cdot b^i = c \cdot \left(\sum_{i=0}^{n} b^i\right) = c \cdot \frac{1 - b^n}{1 - b}.$$

Notice here we start the index at $i = 0$.

Let's introduce some new, convenient notation. If

$$\lim_{n\to\infty} \sum_{i=0}^{n} a_i$$

exists, then we denote it as

$$\sum_{i=0}^{\infty} a_n.$$

Example 1.29. If $|b| < 1$, then $\lim_{n\to\infty} b^n = 0$ from Exercise 1.27. So, from Example 1.28 and limit laws 1.8, if $|b| < 1$, then

$$\lim_{n\to\infty} \sum_{i=0}^{n} c \cdot b^i = \lim_{n\to\infty} c \cdot \frac{1-b^n}{1-b} = c \cdot \frac{1 - \lim\limits_{n\to\infty} b^n}{1-b} = \frac{c}{1-b}.$$

In other words,

$$\sum_{i=0}^{\infty} c \cdot b^i = \frac{c}{1-b}.$$

We now relate decimal notation to fraction notation. Decimal notation is shorthand for summing powers of 10.

Example 1.30. For example, if $c = 3$, $b = \frac{1}{10}$, then

$$\sum_{i=0}^{\infty} \frac{3}{10^i} = \frac{3}{1-\frac{1}{10}} = \frac{10}{3}.$$

Now notice

$$\frac{10}{3} = 1 + \frac{1}{3} = 1.33333... = 1.\overline{3}.$$

$$1.\overline{3} := \sum_{i=0}^{\infty} \frac{3}{10^i}.$$

The left side of the equality is notation. The right side has some mathematical meaning as described in these notes.

Example 1.31. For example, $\frac{1}{9} = 0.\overline{1}$. Indeed,

$$0.\overline{1} := \sum_{i=1}^{\infty} \frac{1}{10^i} = \sum_{i=0}^{\infty} \frac{1}{10^i} - 1 = \frac{1}{1-\frac{1}{10}} - 1 = \frac{1}{9}.$$

Alternatively, $\sum_{i=1}^{\infty} \frac{1}{10^i} = \sum_{i=0}^{\infty} \frac{1}{10^{i+1}} = \sum_{i=0}^{\infty} \frac{10^{-1}}{10^i} = 10^{-1}/(1-1/10) = \frac{1}{9}$.

We note $0.\overline{9} = 1$. This follows from the previous computation, since

$$0.\overline{9} := 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i} = \frac{9}{9} = 1.$$

Exercise 1.32.

(a) Express

$$7 \cdot \sum_{i=0}^{\infty} 10^{-2i+2}$$

as a rational number.

(b) Express the rational number

$$\frac{1}{7}$$

as

$$a \cdot \sum_{i=0}^{\infty} 10^{m(i+1)}$$

for some integers a and m.

(c) Find a and m as in (b) but for $\frac{4}{7}$.

(d) Find a and m as in (b) but for $\frac{1}{13}$.

(e) Express the rational number

$$\frac{1}{12}$$

as

$$b \cdot 10^n + a \cdot \sum_{i=0}^{\infty} 10^{m(i+1)+n}$$

for some integers a, b, m and n.

(f) For any nonzero integer q, express the rational number

$$\frac{1}{q}$$

as

$$b \cdot 10^n + a \cdot \sum_{i=0}^{\infty} 10^{m(i+1)+n}$$

for some integers a, b, m and n. Suggestion: follow these steps.

(i) [Strang, pg 373] There are some negative integers m, n, some integers b, z and r such that $0 \leq r < q$, $10^{-n} = bq + r$ and $10^{-m-n} = zq + r$. (pigeonhole principle: since there are infinitely many positive integer powers of 10, there exists two powers of ten such that division by q of them will yield the same remainder less than q.)

(ii) Define $a = z - b \cdot 10^{-m}$.

(iii)

$$\frac{1}{q} = b \cdot 10^n + a \cdot \sum_{i=0}^{\infty} 10^{m(i+1)+n}$$

Exercise 1.33. The sequence in Examples 1.5 and 1.13: $a_0 = 2$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for any $n \in \mathbb{N}$, converges. Justify the following statements.

(a) For all $n \in \mathbb{N}$, $a_n \geq 1$ and $a_n \leq 2$. *Hint: induction: we can follow the pattern of the last two theorems in Appendix A.*

(b) For all $n \in \mathbb{N}$, $|a_{n+1} - a_n| \leq \frac{1}{2^n}$. *Hint: induction.*

(c) For all $m, n \in \mathbb{N}$, $|a_m - a_n| \leq \sum_{k=n}^{m-1} \frac{1}{2^k}$. *Hint: Repeated use of (b) and the triangle inequality.*

(d) The sequence $\{a_n\}_{n=0}^{\infty}$ converges. *Hint: Theorem B.2 and the Completeness of the Reals*

Exercise 1.34. Approximate the square root of any positive integer with a recursive sequence of rational numbers.

1.2.2 Convergence Tests

Definition 1.35. For a given sequence $\{a_n\}_{n=n_0}^{\infty}$, the series $\{\sum_{k=n_0}^{n} a_k\}_{n=n_0}^{\infty}$ is **absolutely convergent** or **converges absolutely** if the series $\{\sum_{k=n_0}^{n} |a_k|\}_{n=n_0}^{\infty}$ converges.

Exercise 1.36. Justify the following statements.

(a) If $\{a_n\}_{n=n_0}^{\infty}$ is a sequence such that eventually $a_n \geq 0$ and $\{\sum_{k=n_0}^{n} a_k\}_{n=n_0}^{\infty}$ converges, then $\{\sum_{k=n_0}^{n} a_k\}_{n=n_0}^{\infty}$ converges absolutely.

(b) If $\{\sum_{k=n_0}^{n} a_k\}_{n=n_0}^{\infty}$ converges absolutely, then it converges.

Exercise 1.37. Justify the following statement. The series $\{\sum_{k=n_0}^{n} |a_k|\}_{n=n_0}^{\infty}$ is monotone increasing.

Theorem 1.38. (The comparison test)

(a) If $\{\sum_{k=n_0}^{n} b_k\}_{n=n_0}^{\infty}$ converges absolutely and eventually $|a_n| \leq |b_n|$, then $\{\sum_{k=n_0}^{n} a_k\}_{n=n_0}^{\infty}$ converges absolutely.

(b) If $\{\sum_{k=n_0}^{n} |a_k|\}_{n=n_0}^{\infty}$ diverges to ∞ and eventually $|a_n| \leq |b_n|$, then $\{\sum_{k=n_0}^{n} |b_k|\}_{n=n_0}^{\infty}$ diverges to ∞.

Proof. (a) If eventually $|a_n| \leq |b_n|$, then there is some $N \geq n_0$ such that, for all $n \geq N$, $|a_n| \leq |b_n|$. Thus, for all $n \geq N$,

$$\sum_{k=N}^{n} |a_k| \leq \sum_{k=N}^{n} |b_k|.$$

Since $\{\sum_{k=0}^{n} |b_k|\}_{n=n_0}^{\infty}$ is monotone increasing (by the previous exercise) and converges (since the series $\{\sum_{k=n_0}^{n} b_k\}_{n=n_0}^{\infty}$ converges absolutely),

$$\sum_{k=N}^{n} |b_k| \leq \sum_{k=N}^{\infty} |b_k| < \infty$$

for all $n \geq N$. Hence, for all $n \geq n_0$,

$$\sum_{k=n_0}^{n} |a_k| \leq \sum_{k=N}^{\infty} |b_k| + \sum_{k=n_0}^{N-1} |a_k|.$$

In other words, $\{\sum_{k=n_0}^{n} |a_k|\}_{n=n_0}^{\infty}$ is a bounded sequence. Since it's monotone, it converges by Theorem 1.18.

(b) If eventually $|a_n| \leq |b_n|$, then there is some $N_1 \geq n_0$ such that, for all $n \geq N_1$, $|a_n| \leq |b_n|$. Thus, for all $n \geq N_1$,

$$\sum_{k=N_1}^{n} |a_k| \leq \sum_{k=N_1}^{n} |b_k|.$$

Fix $M > 0$. Then there is some $N \geq N_1$ such that

$$\sum_{k=n_0}^{n} |a_k| > M + \sum_{k=n_0}^{N_1-1} |a_k| - \sum_{k=n_0}^{N_1-1} |b_k|$$

for all $n \geq N$ (since $\{\sum_{k=n_0}^{n} |a_k|\}_{n=n_0}^{\infty}$ diverges to ∞). Hence, for all $n \geq N$,

$$\sum_{k=n_0}^{n} |b_k| = \sum_{k=n_0}^{N_1-1} |b_k| + \sum_{k=N_1}^{n} |b_k| \geq \sum_{k=n_0}^{N_1-1} |b_k| + \sum_{k=N_1}^{n} |a_k| > M.$$

Hence, $\{\sum_{k=n_0}^{n} |b_k|\}_{n=n_0}^{\infty}$ diverges to ∞.

□

Theorem 1.39. (The integral test) If f is a continuous function defined on $[1, \infty)$, such that $|f|$ is decreasing, and if $a_n = f(n)$ for all $n \in \mathbb{N}$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely if and only if $\int\limits_{1}^{\infty} |f(x)|dx$ exists.

Proof. The proof is outlined in the next exercise. □

Exercise 1.40. The following steps are adapted from one of Jody Ryker's exercises.

(a) Draw a function that is continuous, positive, and decreasing for $x \geq 1$.

(b) Suppose $a_k = f(k)$ for $k \in \mathbb{N}$. Graph rectangles of unit width and height a_k for $k = 2, 3, ..., n$. These should be "right-endpoint" rectangles. What do you notice about the area under the curve of $f(x)$ for $1 \leq x \leq n$ and the area of these rectangles?

(c) On a new set of axes, draw f once again for $x \geq 1$. Now add rectangles of unit width and height a_k for $k = 1, 2, ..., n-1$. These should be "left-endpoint" rectangles. What do you notice about the area under the curve of $f(x)$ for $1 \leq x \leq n$ and the area of these rectangles?

(d) Use your graphs to construct an inequality statement between $\int_1^n f(x)dx$, $\sum_{k=2}^{n} a_k$, and $\sum_{k=1}^{n-1} a_k$.

(e) Describe the previous inequalities in terms of upper and lower Darboux sums (Appendix C).

(f) Now suppose that $\int_1^\infty f(x)dx = L \in \mathbb{R}$. What's the relationship between $\int_1^n f(x)dx$ and $\int_1^\infty f(x)dx$? Use this relationship and your work from the previous part to show that $\sum_{k=1}^{\infty} a_k$ must also converge.

(g) Now suppose that $\sum_{k=1}^{\infty} a_k = L \in \mathbb{R}$. What's the relationship between $\sum_{k=1}^{n-1} a_k$ and $\sum_{k=1}^{\infty} a_k$? Use this relationship and your previous work to show that $\int_1^\infty f(x)dx$ must also converge.

Exercise 1.41. Suppose $p \in \mathbb{R}$. Justify the following statements.

(a) If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ exists.

(b) If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ does not exist.

The series formed by the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is called the **harmonic series**. The limit of the harmonic series diverges to ∞ by the integral test. This series is, for example, a counterexample of the converse of the following statement.

Exercise 1.42. If $\sum_{n=n_0}^{\infty} a_n$ exists, then $\lim_{n\to\infty} a_n = 0$.

Exercise 1.43. Justify the following statements.

(a) If the series formed by the sequence $\{a_n\}_{n=0}^{\infty}$ converges, then there is a convergent series formed by some sequence $\{b_n\}_{n=0}^{\infty}$ such that $a_n < b_n$ for all $n \in \mathbb{N}$.

(b) If the series formed by the sequence $\{a_n\}_{n=0}^{\infty}$ converges, then there is a convergent series formed by some sequence $\{b_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$.

Theorem 1.44.

(a) (The ratio test) If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, then

- If $L < 1$, then $\{\sum_{k=0}^{n} a_k\}_{n=0}^{\infty}$ converges absolutely.
- If $L > 1$, then $\sum_{n=0}^{\infty} |a_n|$ diverges to ∞.
- If $L = 1$, then the test is inconclusive. There exist sequences with such a limit which converge, and some which don't.

(b) (The root test) If $\lim\limits_{n\to\infty} (|a_n|)^{\frac{1}{n}} = L$, then

- If $L < 1$, then $\{\sum_{k=0}^{n} a_k\}_{n=0}^{\infty}$ converges absolutely.
- If $L > 1$, then $\sum_{n=0}^{\infty} |a_n|$ diverges to ∞.
- If $L = 1$, then the test is inconclusive. There exist sequences with such a limit which converge, and some which don't.

Proof.

(a) If $n \geq N$, we rewrite

$$a_{n+1} = a_n \cdot \frac{a_{n+1}}{a_n} = \cdots = a_N \cdot \prod_{k=N}^{n} \frac{a_{k+1}}{a_k}.$$

Here $\prod_{k=N}^{n} \frac{a_{k+1}}{a_k}$ is defined recursively as $\prod_{k=N}^{N} \frac{a_{k+1}}{a_k} = \frac{a_{N+1}}{a_N}$ and $\prod_{k=N}^{n+1} \frac{a_{k+1}}{a_k} = \frac{a_{n+2}}{a_{n+1}} \cdot \prod_{k=N}^{n} \frac{a_{k+1}}{a_k}$.

- If $L < 1$, then there is some $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $|L + \varepsilon| < 1$ and $|\frac{a_{n+1}}{a_n}| < L + \varepsilon$ for all $n \geq N$.
  Hence, if $n \geq N$,
$$|a_{n+1}| = |a_N| \cdot \prod_{k=N}^{n} \left|\frac{a_{k+1}}{a_k}\right| < |a_N| \cdot (L+\varepsilon)^{n-N+1} = |a_N| \cdot (L+\varepsilon)^{-N} \cdot (L+\varepsilon)^{n+1}.$$

Now we apply the comparison test to compare $|a_n|$ with $|a_N| \cdot (L+\varepsilon)^{-N} \cdot (L+\varepsilon)^n$. The latter is a geometric sequence $a \cdot b^n$ with $a = |a_N| \cdot (L+\varepsilon)^{-N}$ and $|b| = |L+\varepsilon| < 1$. Hence, $\{\sum_{k=0}^{n} a_k\}_{n=0}^{\infty}$ converges absolutely if $L < 1$.

- If $L > 1$, we use part (b) of the comparison test. There is some $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $L - \varepsilon > 1$ and $L - \varepsilon < |\frac{a_{n+1}}{a_n}|$ for all $n \geq N$. Hence,

$$|a_{n+1}| > |a_N| \cdot (L-\varepsilon)^{-N} \cdot (L-\varepsilon)^{n+1}$$

for all $n \geq N$. And since $L - \varepsilon > 1$, the series $\{|a_N| \cdot (L-\varepsilon)^{-N} \cdot \sum_{k=0}^{n}(L-\varepsilon)^n\}_{n=0}^{\infty}$ diverges to ∞. Hence, by (b) of the comparison test, the series $\{\sum_{k=0}^{n} |a_k|\}_{n=0}^{\infty}$ diverges to ∞.

- For example, the harmonic series diverges and $|\frac{n}{n+1}| \to 1$ as $n \to \infty$. But the series formed by $\frac{1}{n^2}$ converges, and $\frac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$. Hence, $L = 1$ gives us no information about the convergence of the series.

□

Exercise 1.45. The root test can be proved in a way similar to the way the ratio test was proved in the previous theorem - by comparison with an appropriate geometric series. Fill in the details.

Exercise 1.46. For the following sequences, use an appropriate test to determine which of the corresponding series converge.

(a) $\frac{4}{n}$

(b) $\frac{1}{\sqrt{n^2+1}}$

(c) $\frac{(-1)^n}{n^2}$

(d) $\frac{3n-14}{n^2}$

(e) $\frac{1}{2^{2n}}$

(f) $\frac{n}{e^n}$

(g) $\frac{n}{n^n}$ [Strang, pg 379]

(h) $m > 1$, $\frac{n}{m^n}$

(i) $x \in \mathbb{R}$, $p \in \mathbb{R}$, $\frac{x^n}{n^p}$. [Strang, pg 378]

(j) $x \in \mathbb{R}$, $\frac{x}{n!}$ [Strang, pg 378]

(k) $p \in \mathbb{R}$, $\frac{1}{(\ln(n))^p}$ [Strang, pg 381]

1.2.3 Conditional Convergence

Definition 1.47. A series converges **conditionally** if it converges but does not converge absolutely. Compare Exercise 1.36.

Exercise 1.48. From [Strang, pg 383]. If $0 \leq a_{n+1} \leq a_n$ for all n, and $\lim_{n\to\infty} a_n = 0$, then the **alternating series** formed by $\{(-1)^{n+1}a_n\}_{n=1}^{\infty}$ converges. Proof: Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$. Justify each of the following assertions.

(a) $s_{2n+1} \leq s_{2n-1}$ for all n.

(b) $s_{2n-2} \leq s_{2n}$ for all n.

(c) $s_{2n} \leq s_1$ for all n.

(d) $s_{2n+1} \geq s_2$ for all n.

(e) $\{s_{2n+1}\}_{n=1}^{\infty}$ is a bounded monotone decreasing sequence.

(f) $\{s_{2n}\}_{n=1}^{\infty}$ is a bounded monotone increasing sequence.

(g) $\{s_{2n+1}\}_{n=1}^{\infty}$ converges.

(h) $\{s_{2n}\}_{n=1}^{\infty}$ converges.

(i) $|s_{n-1} - s_n| = a_n$.

(j) $\{s_{2n+1}\}_{n=1}^{\infty}$ and $\{s_{2n}\}_{n=1}^{\infty}$ converge to the same limit.

(k) $\{s_n\}_{n=1}^{\infty}$ converges.

Exercise 1.49. Adapted from [Strang, Exercise 45, pg 381]. The alternating series formed by $\{(-1)^{n+1}\frac{1}{n}\}_{n=1}^{\infty}$ converges conditionally to $\ln(2)$. Proof: Justify each of the following assertions.

(a)

$$\sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k} = \sum_{k=n+1}^{2n}\frac{1}{k}.$$

Hint: add $2 \cdot \sum_{k=1}^{n}\frac{1}{2k}$ *to both sides.*

(b)

$$\int_{n+1}^{2n+1}\frac{1}{x}dx \leq \sum_{k=n+1}^{2n}\frac{1}{k} \leq \int_{n}^{2n}\frac{1}{x}dx.$$

Hint: Exercise 1.40.

(c)

$$\lim_{n\to\infty}\sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k} = \ln(2)$$

(d) The alternating series formed by $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to $\ln(2)$. *Hint: The previous Exercise and Theorem 1.21.*

(e) The series formed by $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ does not converge absolutely.

(f) Hence, the series conditionally converges to $\ln(2)$.

Exercise 1.50. (The Riemann Rearrangement Theorem) If the series formed by $\{a_n\}_{n=0}^{\infty}$ is conditionally convergent, then, for any $M \in \mathbb{R}$, there is some bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the series formed by $\{a_{\sigma(n)}\}_{n=0}^{\infty}$ converges to M.

Exercise 1.51. From [Strang, Exercise 10.3.30] Explain why

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

Exercise 1.52. Determine if the following are conditionally convergent or absolutely convergent. Justify using an appropriate test or theorem. The series formed by

(a) $\{(-2)^{-n}\}_{n=0}^{\infty}$

(b) $\{\frac{\cos(n\pi)}{n^{\frac{3}{2}}}\}_{n=0}^{\infty}$

(c) $\{\frac{(-1)^n 2^n}{n!}\}_{n=0}^{\infty}$

(d) $\{\frac{(-1)^n n^n}{5^n}\}_{n=0}^{\infty}$

1.2.4 Power Series

Definition 1.53. The **domain of the power series** centered at $a \in \mathbb{R}$ and formed by the sequence $\{a_n\}_{n=0}^{\infty}$ is the set of all x such that $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ converges. The **power series** centered at $a \in \mathbb{R}$ and formed by the sequence $\{a_n\}_{n=0}^{\infty}$ is the real-valued function with domain the domain of the power series centered at $a \in \mathbb{R}$ and formed by the sequence $\{a_n\}_{n=0}^{\infty}$ and with output $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ for every x in its domain.

In particular, every series is a power series centered at a and evaluated at $a+1$.

Theorem 1.54. The power series centered at $a \in \mathbb{R}$ and formed by the sequence $\{a_n\}_{n=0}^{\infty}$ has domain either $\{a\}$, $\mathbb{R}$, or $(a-R, a+R)$ for some $R \in (0, \infty)$. R in this case is called the **radius of convergence** of the power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ and can also be either 0 or ∞.

Proof. Adapted from [Strang, pg 391]. We define

$$R = \sup\left\{ |x-a| \,\middle|\, \left\{ \sum_{k=0}^{n} a_k(x-a)^k \right\}_{n=0}^{\infty} \text{ converges} \right\}$$

- Suppose first $0 < R < \infty$. Suppose further $0 < |x-a| < R$. Then by the Exercise B.5, there is some $x_0 \in \mathbb{R}$ such that $|x-a| < |x_0 - a| < R$ and $\left\{\sum_{k=0}^{n} a_k(x_0-a)^k\right\}_{n=0}^{\infty}$ converges. Define $r = x_0 - a$. Since $\left\{\sum_{k=0}^{n} a_k r^k\right\}_{n=0}^{\infty}$ converges, there is some $N \in \mathbb{N}$ such that $|a_n r^n| < 1$ for all $n \geq N$ by Exercise 1.42. If $|x-a| < |r|$, then $|a_n(x-a)^n| = |\frac{(x-a)^n}{r^n}||a_n r^n| < |\frac{x-a}{r}|^n$ for all $n \geq N$. Hence, since $\frac{|x-a|}{|r|} < 1$, by comparison with the geometric series formed by $\left\{\left(\frac{|x-a|}{|r|}\right)^n\right\}_{n=0}^{\infty}$, the series $\left\{\sum_{k=0}^{n} a_k(x-a)^k\right\}_{n=0}^{\infty}$ converges absolutely.
- If $R = \infty$, then every $x \in \mathbb{R}$ is in the domain of this power series by the previous argument.
- Certainly $R \geq 0$ since $\sum_{k=0}^{\infty} a_k \cdot 0^k = a_0$. If $R = 0$, then a is the only number in the domain of this power series.

□

Exercise 1.55. The radius of convergence of a power series formed by $\{a_n\}_{n=0}^{\infty}$ is the reciprocal of the number $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ if it exists and is nonzero. Explain.

Exercise 1.56. For each of the following power series, find its radius of convergence. Justify.

(a) Centered at 0 and formed by $\{n\}_{n=1}^{\infty}$.

(b) Centered at 0 and formed by $\{n!\}_{n=0}^{\infty}$.

(c) Centered at 0 and formed by $\{\frac{1}{n^p}\}_{n=1}^{\infty}$ if $p \in \mathbb{R}$.

(d) Centered at 0 and formed by $\{a_n\}_{n=0}^{\infty}$ if $a_{4n} = 1$, $a_{2n+1} = 0$, and $a_{4n+2} = -1$ for all $n \in \mathbb{N}$.

(e) Centered at 0 and formed by $\{b^n\}_{n=0}^{\infty}$ if $b \in \mathbb{R}$.

(f) Centered at $a \in \mathbb{R}$ and formed by $\{\frac{1}{n!}\}_{n=0}^{\infty}$.

(g) Centered at $a \in \mathbb{R}$ and formed by $\{\frac{(-1)^n}{(2n+1)!}\}_{n=0}^{\infty}$.

(h) Centered at $a \in \mathbb{R}$ and formed by $\{\frac{(-1)^n}{(2n)!}\}_{n=0}^{\infty}$.

Exercise 1.57.

(a) Find a power series centered at 0 which doesn't converge when evaluated at its radius of convergence but does converges at the negative of its radius of convergence.

(b) Find a power series centered at 0 which converges when evaluated at both its radius of convergence and the negative of its radius of convergence. How is this possible?

Exercise 1.58. Suppose $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge. Explain.

1. $\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$ converges and equals $\sum_{n=0}^{\infty} (a_n + b_n) x^n$.
2. $\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n$ converges and equals $\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$.

Exercise 1.59. Suppose f is the power series centered at $a \in \mathbb{R}$ and formed by the sequence $\{a_n\}_{n=0}^{\infty}$. Explain.

(a) f is integrable on $[a, x]$ for all x in the domain of f. Moreover,

$$\int_a^x f(s)ds = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}.$$

(b) f is infinitely differentiable in its domain. Moreover, for all $k \in \mathbb{N}$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x-a)^{n-k}$$

for all x in the domain of f.

1.2.5 Taylor Series

A power series is a certain type of function. An infinitely differentiable function has a Taylor series at every point, which is a power series whose coefficients depend on the derivatives of the function (and whose radius of convergence might be 0).

Definition 1.60. For an infinitely differentiable function, f, for any a in the domain of f, we define the **Taylor series** of f centered at a to be the power series centered at a and formed by $\left\{ \frac{f^{(n)}(a)}{n!} \right\}_{n=0}^{\infty}$.

Exercise 1.59 (a) is a sort of converse: A power series f centered at a and formed by $\{a_n\}_{n=0}^{\infty}$ satisfies $f^{(n)}(a) = n! a_n$. In other words, the Taylor series of a power series is the power series itself.

Exercise 1.61. From [Strang, pg 394]

(a) Find the Taylor series of

(i)

$$\frac{1}{1-x}$$

(ii)

$$\frac{1}{1+x^2}$$

(iii)

$$-\ln(1-x)$$

(b) Find relationships between these three functions.

Exercise 1.62.

(a) Assuming the solution of the initial value problem

$$\begin{cases} y'(x) = y(x) & \text{if} \quad x \in \mathbb{R} \\ y(0) = 1 \end{cases}$$

is differentiable, explain why it's infinitely differentiable and find its Taylor series centered at 0.

(b) Assuming the solution of the initial value problem

$$\begin{cases} y''(x) = -y(x) & \text{if} \quad x \in \mathbb{R} \\ y'(0) = 1 \\ y(0) = 0 \end{cases}$$

is twice differentiable, explain why it's infinitely differentiable and find its Taylor series centered at 0.

(c) Assuming the solution of the initial value problem

$$\begin{cases} y''(x) = -y(x) & \text{if} \quad x \in \mathbb{R} \\ y'(0) = 0 \\ y(0) = 1 \end{cases}$$

is twice differentiable, explain why it's infinitely differentiable and find its Taylor series centered at 0.

Compare with the last three parts of Exercise 1.59. What are other names for these functions?

Exercise 1.63. Define

$$f(x) = \begin{cases} e^{-1/x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

(a) Explain why f is infinitely differentiable on $\mathbb{R}$.

(b) Find its Taylor series centered at 0.

(c) Where does f agree with its Taylor series centered at 0?

Exercise 1.64. (Generalized Mean Value Theorem) If $n \in \mathbb{N}$, $a \neq x \in \mathbb{R}$, and f is a $n+1$ differentiable real-valued function defined on an open interval containing a and x, then for some c in between a and x,

$$f(x) - T_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

if $T_{n,a}$ is the nth partial sum of f's Taylor series: $T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. Adapted from [Strang, pg 391 and 10.5.23] Prove the Generalized Mean Value Theorem by explaining each step.

(a) If $g(x) = (x-a)^{n+1}$ and $k \in \{0, \ldots, n+1\}$, then $g^{(k)}(b) \neq 0$ for any $b \neq a$.

(b) $g^{(k)}(a) = 0$ for all $k \in \mathbb{N}$ except for $k = n+1$.

(c) $g^{(n+1)}(x) = (n+1)!$ for all $x \in \mathbb{R}$.

(d) If $R_n = f - T_{n,a}$ and $k \in \{0, \ldots, n\}$, then $R_n^{(k)}(a) = 0$.

(e) $R_n^{(n+1)}(x) = f^{(n+1)}(x)$ for all x in the domain of f.

(f) If $c_0 = x$ and $k \in \{0, \ldots, n\}$, then there are numbers $c_1, \ldots, c_{n+1}$ such that c_{k+1} is strictly between a and c_k and

$$\frac{R_n^{(k)}(c_k)}{g^{(k)}(c_k)} = \frac{R_n^{(k+1)}(c_{k+1})}{g^{(k+1)}(c_{k+1})}.$$

Hint: Use the Cauchy Mean Value Theorem.

(g) If $c = c_{n+1}$, then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

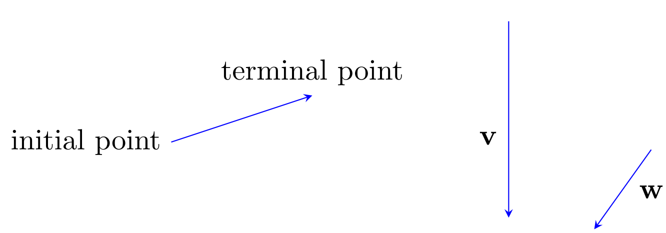
---

2 Vectors

2.1 Vectors Geometrically

Definition 2.1. A **vector** is a finite line segment whose endpoints are ordered. One endpoint is called the **initial point**, while the other is called the **terminal point**. That is, vectors are line segments with a preferred direction. When drawn, the terminal point is distinguished by an arrowhead.

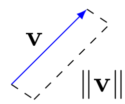
Example 2.2. Vectors are denoted with boldface lowercase letters.



□

Definition 2.3. The **magnitude** or **length** of a vector, $\mathbf{v}$, is the length of the line segment corresponding to $\mathbf{v}$, denoted by $\|\mathbf{v}\|$.

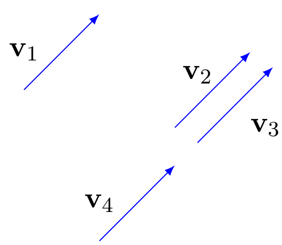
Example 2.4.



□

Definition 2.5. Two vectors are **equivalent** if they have the same magnitude and direction. If $\mathbf{v}$ and $\mathbf{w}$ are equivalent, we write $\mathbf{v} = \mathbf{w}$.

Example 2.6. Below, $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4$.



□

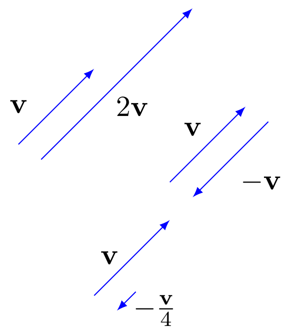
Definition 2.7. The **zero vector**, $\mathbf{0}$, is the vector whose initial and terminal points are the same.

Definition 2.8. If $k \in \mathbb{R}$ ($\in$ denotes membership), that is, if k is a real number, otherwise known as a **scalar**, and $\mathbf{v}$ is a vector in the plane, then we may define a new vector in the plane, $k\mathbf{v}$, as follows:

- If $k > 0$, $k\mathbf{v}$ is the unique vector (up to equivalence) with the same initial point as $\mathbf{v}$, but with magnitude $k\|\mathbf{v}\|$.
- If $k = -1$, denote $(-1)\mathbf{v}$ by $-\mathbf{v}$ and define $-\mathbf{v}$ as the unique vector whose initial point is the terminal point of $\mathbf{v}$, and vice versa.
- If $k < 0$, define $k\mathbf{v} := -k(-\mathbf{v})$.
- If $k = 0$, $k\mathbf{v} := \mathbf{0}$.

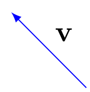
The operation sending k and $\mathbf{v}$ to $k\mathbf{v}$ is called **scalar multiplication**. The notation $A := B$ means we're defining object A to be object B. That is, we know what B is, and we're defining A to be that object.

Example 2.9.



□

Exercise 2.10. Given



find

(a) $-\mathbf{v}$

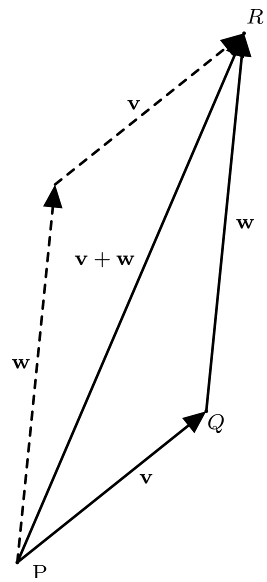
(b) $2\mathbf{v}$

(c) $\frac{\mathbf{v}}{2}$

(d) $-3\mathbf{v}$.

Definition 2.11. If P is the initial point of $\mathbf{v}$, Q its terminal point, we commonly write $\mathbf{v} = \mathbf{PQ}$, and $\mathbf{v}$ is known as the **displacement** from P to Q.

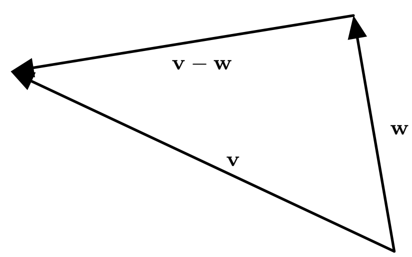
Definition 2.12. Vector Addition. If $\mathbf{v}$ and $\mathbf{w}$ are vectors in the plane, then we may define the sum, $\mathbf{v} + \mathbf{w}$ as follows. If $\mathbf{v} = \mathbf{PQ}$ for some points P and Q, then we can write $\mathbf{w} = \mathbf{QR}$ for some point R. Then we define $\mathbf{v} + \mathbf{w} := \mathbf{PR}$. In words, if the initial point of $\mathbf{w}$ is placed at the terminal point of $\mathbf{v}$, then $\mathbf{v} + \mathbf{w}$ is the vector with $\mathbf{v}$'s initial point and $\mathbf{w}$'s terminal point.



This image also shows $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

Exercise 2.13. From [YF, pg 29] A spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, and then 280 m at 30° east of north. After a fourth unmeasured displacement, she finds herself back where she started. Use a scale drawing to determine the magnitude and direction of the fourth displacement.

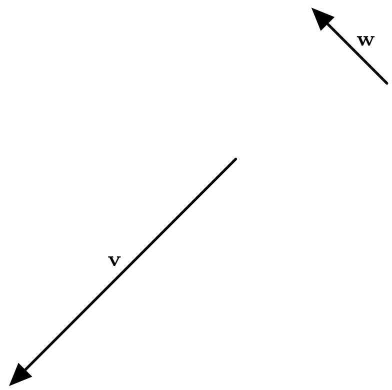
Definition 2.14. The **vector difference** of $\mathbf{v}$ and $\mathbf{w}$ is denoted as $\mathbf{v} - \mathbf{w}$ and defined to be the unique vector $\mathbf{z}$ such that $\mathbf{w} + \mathbf{z} = \mathbf{v}$:



That is, if the initial points of $\mathbf{v}$ and $\mathbf{w}$ are the same, then $\mathbf{v} - \mathbf{w}$ is the vector with initial point the terminal point of $\mathbf{w}$, and terminal point the terminal point of $\mathbf{v}$. Rather, if $\mathbf{v} = \mathbf{PQ}$ and $\mathbf{w} = \mathbf{PR}$, then $\mathbf{v} - \mathbf{w} := \mathbf{RQ}$.

Remark 2.15. We note $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$.

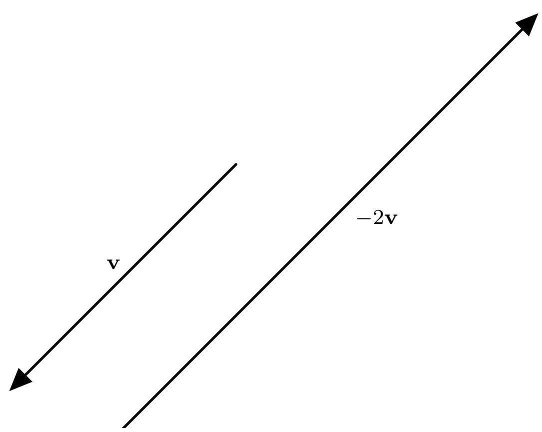
Example 2.16. Given



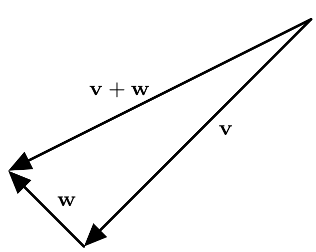
find

1. $-2\mathbf{v}$.
2. $\mathbf{v}+\mathbf{w}$.
3. $\mathbf{v}-2\mathbf{w}$.

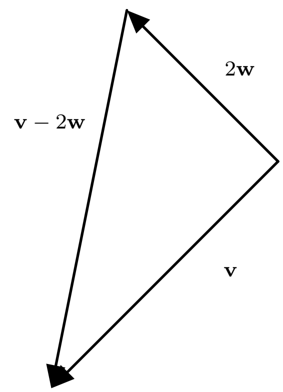
1.



2.



3.



□

Exercise 2.17. Draw two vectors, label one $\mathbf{v}$ and the other $\mathbf{w}$.

(a) Draw $\mathbf{v} + \mathbf{w}$.

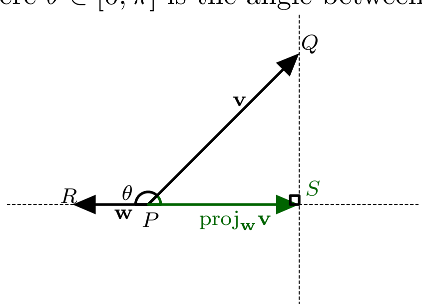
(b) Draw $-2\mathbf{v} + \mathbf{w}$.

(c) Draw $\mathbf{v} - 3\mathbf{w}$.

Definition 2.18. Suppose we have two vectors, $\mathbf{v} = \mathbf{PQ}$ and $\mathbf{w} = \mathbf{PR}$, whose initial points are the same. Then drop a perpendicular from Q to the line which contains the line segment corresponding to $\mathbf{w}$, intersecting at say S. Then define $\operatorname{proj}_{\mathbf{w}} \mathbf{v} := \mathbf{PS}$ ($\operatorname{proj}_{\mathbf{w}} \mathbf{v}$ is read: the projection of $\mathbf{v}$ onto $\mathbf{w}$). Notice then that

$$\|\operatorname{proj}_{\mathbf{w}} \mathbf{v}\| = \|\mathbf{v}\| \, |\cos\theta|$$

where $\theta \in [0, \pi]$ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Note the figure below.



Definition 2.19. The **dot product** of two vectors in the plane, $\mathbf{v}$ and $\mathbf{w}$, is given by,

$$\mathbf{v} \cdot \mathbf{w} := \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta,$$

where $\theta \in [0, \pi]$ is the angle between $\mathbf{v}$ and $\mathbf{w}$ if their initial points are the same.

Proposition 2.20. (Properties of the dot product.)

(a)
$$(\mathbf{v}+\mathbf{z})\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w}+\mathbf{z}\cdot\mathbf{w}. \tag{2.21}$$

(b) For any $k \in \mathbb{R}$,
$$(k\mathbf{v})\cdot\mathbf{w} = k(\mathbf{v}\cdot\mathbf{w}). \tag{2.22}$$

(c)
$$\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}, \tag{2.23}$$

(d)
$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2. \tag{2.24}$$

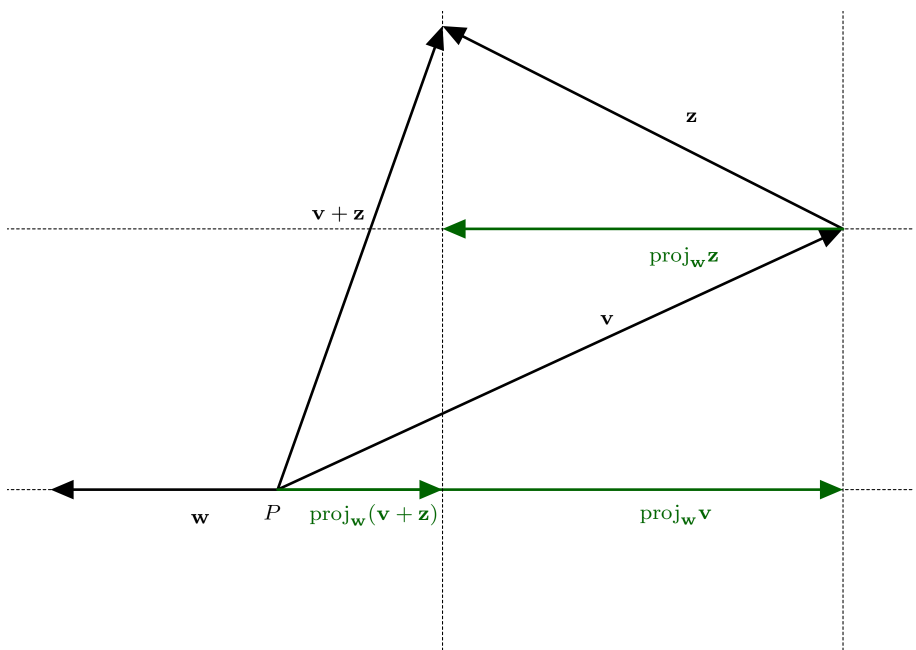
(e)
$$\mathbf{v}\cdot\mathbf{w} = 0 \tag{2.25}$$

if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal (perpendicular).

(f) (Cauchy-Schwarz inequality)
$$|\mathbf{v}\cdot\mathbf{w}| \leq \|\mathbf{v}\|\cdot\|\mathbf{w}\|. \tag{2.26}$$

Proof. The below image shows an example of the fact that $\text{proj}_{\mathbf{w}}(\mathbf{v}+\mathbf{z}) = \text{proj}_{\mathbf{w}}\,\mathbf{v}+\text{proj}_{\mathbf{w}}\,\mathbf{z}$ (notice the initial point of $\text{proj}_{\mathbf{w}}\,\mathbf{v}$ is P). In this case, we have

$$\|\text{proj}_{\mathbf{w}}(\mathbf{v}+\mathbf{z})\| = \|\text{proj}_{\mathbf{w}}\,\mathbf{v}\| - \|\text{proj}_{\mathbf{w}}\,\mathbf{z}\|. \tag{2.27}$$



We have
$$|\mathbf{v}\cdot\mathbf{w}| = \|\text{proj}_{\mathbf{w}}\,\mathbf{v}\|\,\|\mathbf{w}\|$$

in general, with the sign depending on the angle between $\mathbf{v}$ and $\mathbf{w}$. In the above image, we have

$$\mathbf{v} \cdot \mathbf{w} = -\left\|\text{proj}_{\mathbf{w}} \mathbf{v}\right\| \left\|\mathbf{w}\right\|,$$

$$\mathbf{z} \cdot \mathbf{w} = \left\|\text{proj}_{\mathbf{w}} \mathbf{z}\right\| \left\|\mathbf{w}\right\|,$$

and

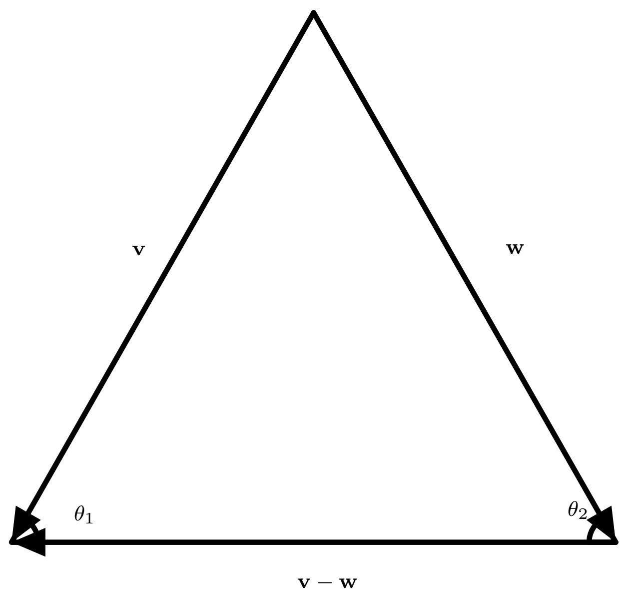
$$(\mathbf{v}+\mathbf{z}) \cdot \mathbf{w} = -\left\|\text{proj}_{\mathbf{w}}(\mathbf{v}+\mathbf{z})\right\| \left\|\mathbf{w}\right\|,$$

which, combined with (2.27), gives (2.21). □

Exercise 2.28. Prove the rest of the properties in the previous proposition.

Example 2.29. Use vectors to show the angles opposite the sides of same length in an isosceles triangle are the same.

Proof. Suppose $\mathbf{v}$ and $\mathbf{w}$ have the same length. That is, suppose $\|\mathbf{v}\| = \|\mathbf{w}\|$. Then $\mathbf{v}, \mathbf{w}$ and **v-w** form an isosceles triangle. From the image:



we have

$$\mathbf{v} \cdot (\mathbf{v} \text{ - } \mathbf{w}) = \|\mathbf{v}\| \, \|\mathbf{v} \text{ - } \mathbf{w}\| \cos\theta_1,$$

and

$$-\mathbf{w} \cdot (\mathbf{v} - \mathbf{w}) = \|-\mathbf{w}\| \, \|\mathbf{v} \text{ - } \mathbf{w}\| \cos\theta_2.$$

And recall that cos is one-to-one in $[0, \pi]$.

□

Exercise 2.30.

(a) Use properties of the dot product to prove

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2$$

for any two vectors $\mathbf{v}$ and $\mathbf{w}$.

(b) State and prove the Pythagorean Theorem using vectors.

Exercise 2.31. Prove the angles in a triangle add up to π.

Definition 2.32. The **distance** between two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as

$$\mathrm{dist}(\mathbf{v},\mathbf{w}) = \|\mathbf{v}-\mathbf{w}\|.$$

Exercise 2.33. Explain, if $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ are vectors:

(a) (symmetry)

$$\mathrm{dist}(\mathbf{v},\mathbf{w}) = \mathrm{dist}(\mathbf{w},\mathbf{v})$$

(b) (indiscernibility)

$$\mathrm{dist}(\mathbf{v},\mathbf{w}) = 0 \text{ if and only if } \mathbf{v}=\mathbf{w}$$

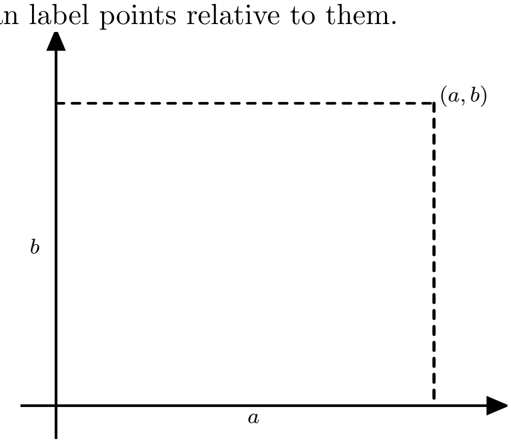
(c) (triangle inequality)

$$\mathrm{dist}(\mathbf{v},\mathbf{w}) \leq \mathrm{dist}(\mathbf{v},\mathbf{x}) + \mathrm{dist}(\mathbf{x},\mathbf{w})$$

Exercise 2.34. Explain why, for any two vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathrm{proj}_{\mathbf{w}}\mathbf{v}$ is the vector closest to $\mathbf{v}$ (with respect to the distance function defined above) contained in the line containing $\mathbf{w}$.

2.2 Vectors Algebraically

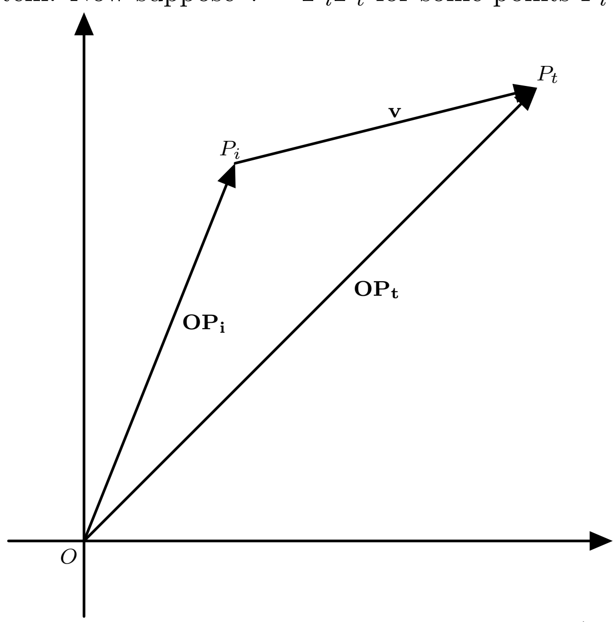
Definition 2.35. Now fix a pair of perpendicular (Cartesian) coordinate axes in the plane so we can label points relative to them.



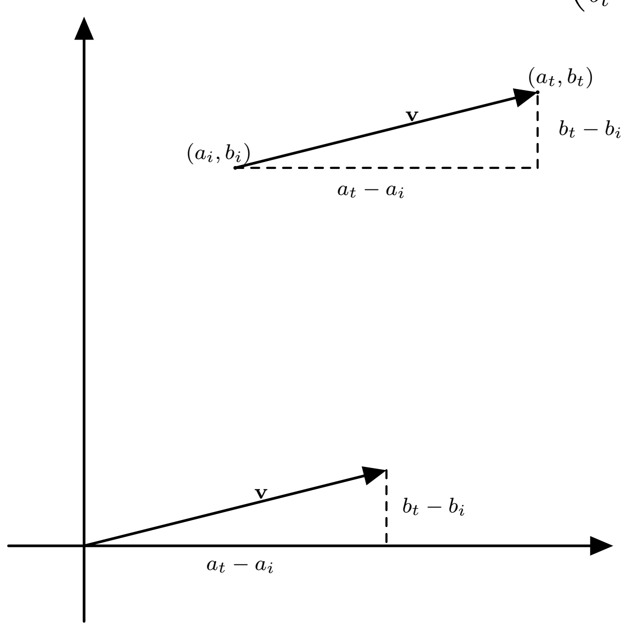
Denote by O the origin. Then, if $P = (a, b)$, denote

$$\begin{pmatrix} a \\ b \end{pmatrix} := \mathbf{OP}.$$

We say the scalars a, b are the **components** of **OP** and **OP** is the **position vector** of P. If $\mathbf{v} = \mathbf{OP}$ for some point P, we say $\mathbf{v}$ is in **standard position** relative to this fixed coordinate system. Now suppose $\mathbf{v} = \mathbf{P}_i\mathbf{P}_t$ for some points $P_i = (a_i, b_i)$ and $P_t = (a_t, b_t)$.



We see that $\mathbf{v} = \mathbf{OP}_t - \mathbf{OP}_i$. But also $\mathbf{v} = \begin{pmatrix} a_t - a_i \\ b_t - b_i \end{pmatrix}$, since



That is,

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} - \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} a_t - a_i \\ b_t - b_i \end{pmatrix}. \tag{2.36}$$

The length of vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is, by the Pythagorean Theorem,

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \sqrt{a^2 + b^2} \tag{2.37}$$

From (2.37) it follows that

$$k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} \tag{2.38}$$

for any scalar k.

As a consequence of (2.36) and (2.38), for any two vectors in the plane in standard position, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, we have

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}.$$

We gather these results in a proposition.

Proposition 2.39. The coordinate description of vectors in the plane. For any scalars, $a, b, k, v_1, v_2, w_1, w_2$,

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \sqrt{a^2 + b^2}$$

$$k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}. \tag{2.40}$$

Definition 2.41. In physics, one might come across such notation as $\begin{pmatrix} a \\ b \end{pmatrix} = a\mathbf{i}+b\mathbf{j}$, where $\mathbf{i} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\mathbf{i}$ and $\mathbf{j}$ are called the **standard unit vectors**. A **unit vector**, $\mathbf{u}$, is any vector with magnitude 1: $\|\mathbf{u}\| = 1$.

Proposition 2.42. For any scalars v_1, v_2, w_1, w_2,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2. \tag{2.43}$$

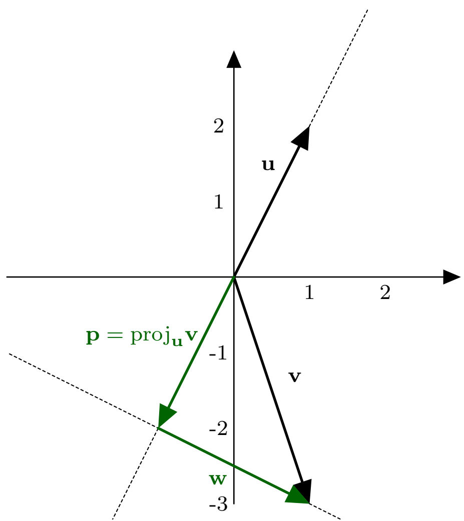
Proof. This follows from (2.21) and (2.22) and the fact that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$, since we write, for example, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j}$. ☐

Exercise 2.44.

(a) Explain why the set of all unit vectors in the plane form a circle.

(b) Write any point $P = (a, b)$ in polar coordinates.

Example 2.45. Given $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, write $\mathbf{v} = \mathbf{p} + \mathbf{w}$, where $\mathbf{p} = \text{proj}_{\mathbf{u}}\,\mathbf{v}$ and $\mathbf{w} = \mathbf{v} - \mathbf{p}$.



Notice

$$\mathbf{p} = \text{proj}_{\mathbf{u}}\,\mathbf{v} = -\left\|\text{proj}_{\mathbf{u}}\,\mathbf{v}\right\| \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}$$

from the fact that

$$\mathbf{v}\cdot\mathbf{u} = -\left\|\text{proj}_{\mathbf{u}}\,\mathbf{v}\right\| \|\mathbf{u}\|$$

in this case.

After some computation, using Proposition 2.42, we obtain $\mathbf{p} = -\mathbf{u}$ and so $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. ☐

Exercise 2.46. Given vectors $\mathbf{v}$ and $\mathbf{u}$ in the plane, find vectors $\mathbf{p}$ and $\mathbf{w}$ such that $\mathbf{p} = k\mathbf{u}$ for some $k \in \mathbb{R}$, $\mathbf{w}$ is perpendicular to $\mathbf{p}$, and $\mathbf{v} = \mathbf{p} + \mathbf{w}$.

Exercise 2.47. From [**OS** Exercise 2.54] A boat is traveling in the water at 30 mph in a direction of N20E (20 degrees East of North). A current is moving at 15 mph in a direction of N45E. What are the new speed and direction of the boat?

2.2.1 Vectors in three dimensions

The definitions and examples for vectors in the plane essentially carry over verbatim to vectors in 3-dimensional space, where we informally describe 3-dimensional space as a plane with an added direction. Formally, we make use of coordinates, where in 3-dimensional space we have 3 axes, all mutually perpendicular.

Definition 2.48.

$$\mathbb{R}^2 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} = \{\text{set of vectors in the plane}\}$$

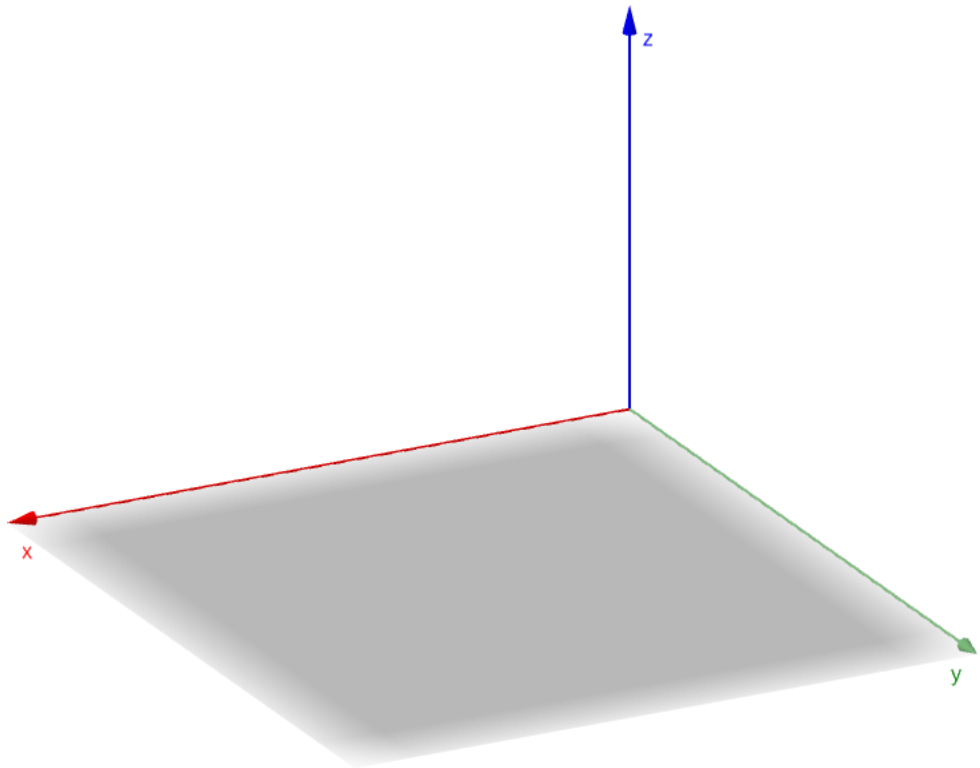
$$\mathbb{R}^3 := \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} = \{\text{set of vectors in 3-dimensional space}\}$$

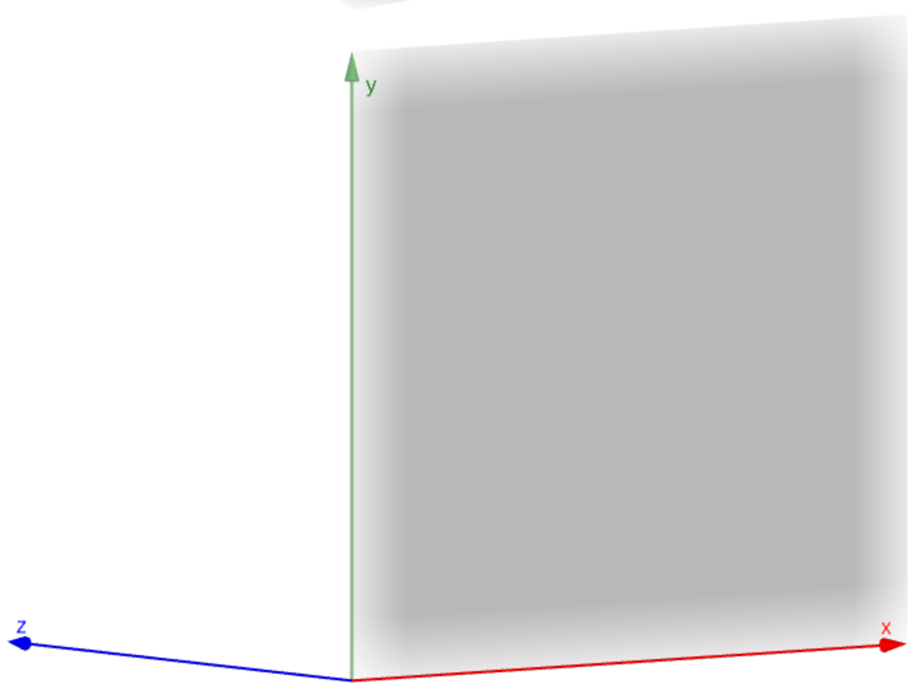
Note: sometimes we will describe $\mathbb{R}^2$ or $\mathbb{R}^3$ as a set of points, not vectors, but this is o.k., due to the one-to-one correspondence, e.g., $P = (a, b) \leftrightarrow \mathbf{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$ (once we fix an origin). Also, $\mathbb{R}^2$ is usually identified as a subset of $\mathbb{R}^3$ as $\left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$, called the xy-plane. More on this later.

Sometimes we write $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and are called the **standard unit vectors**.

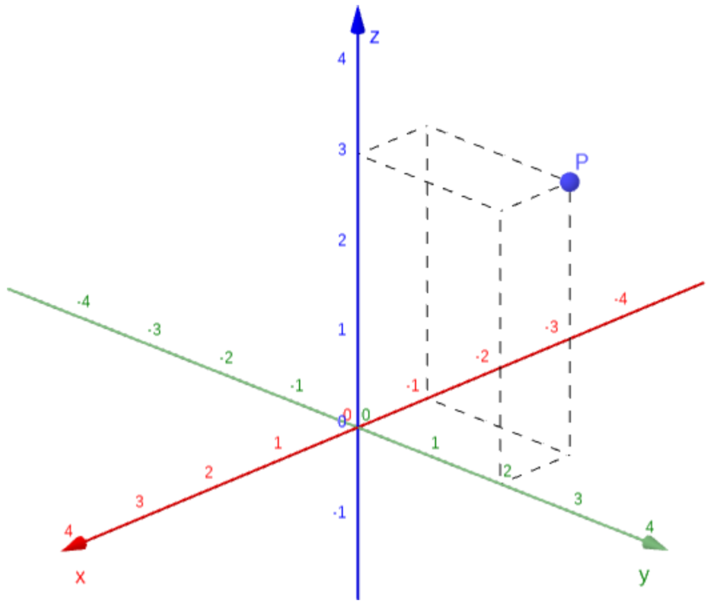
$\mathbf{0} := 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ is called the **zero vector** or the **origin**.

To draw a set of coordinate axes for $\mathbb{R}^3$, we follow the **right-hand rule**: If we take our right hand and point our fingers in the direction of the positive x-axis, and curl our fingers in the direction of the positive y-axis, then the positive z-axis points in the direction of our thumb:





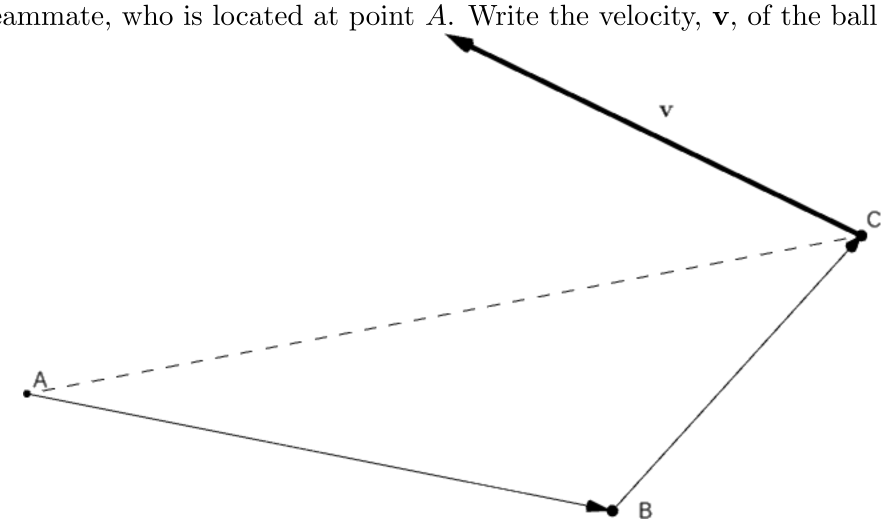
Example 2.49. Sketch the point $P = (-1, 2, 3)$ in 3-dimensional space.



□

Exercise 2.50. The magnitude of $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ is given by $\left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2}$. *Hint: use the Pythagorean Theorem twice.*

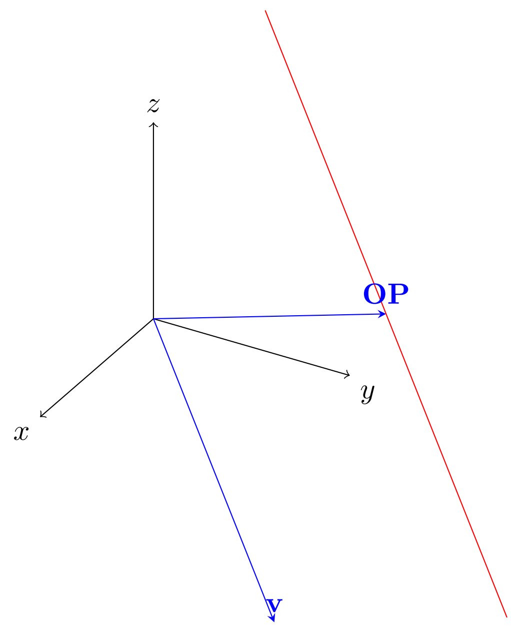
Exercise 2.51. From [**OS** Exercise 2.120] Two soccer players are practicing for a match. One of them runs 10 m from point A to point B. She then turns left at 90° and runs 10 m until she reaches point C. Then she kicks the ball with a speed of 10 m/s at an upward angle of 45° to her teammate, who is located at point A. Write the velocity, $\mathbf{v}$, of the ball in component form.



2.2.2 Planes in $\mathbb{R}^3$

Definition 2.52. Fix a point $P \in \mathbb{R}^3$ and a vector $\mathbf{v} \in \mathbb{R}^3$. The **line** containing P in the direction of $\mathbf{v}$ is the set of points

$$\{s\mathbf{v} + \mathbf{OP} \mid s \in \mathbb{R}\}$$



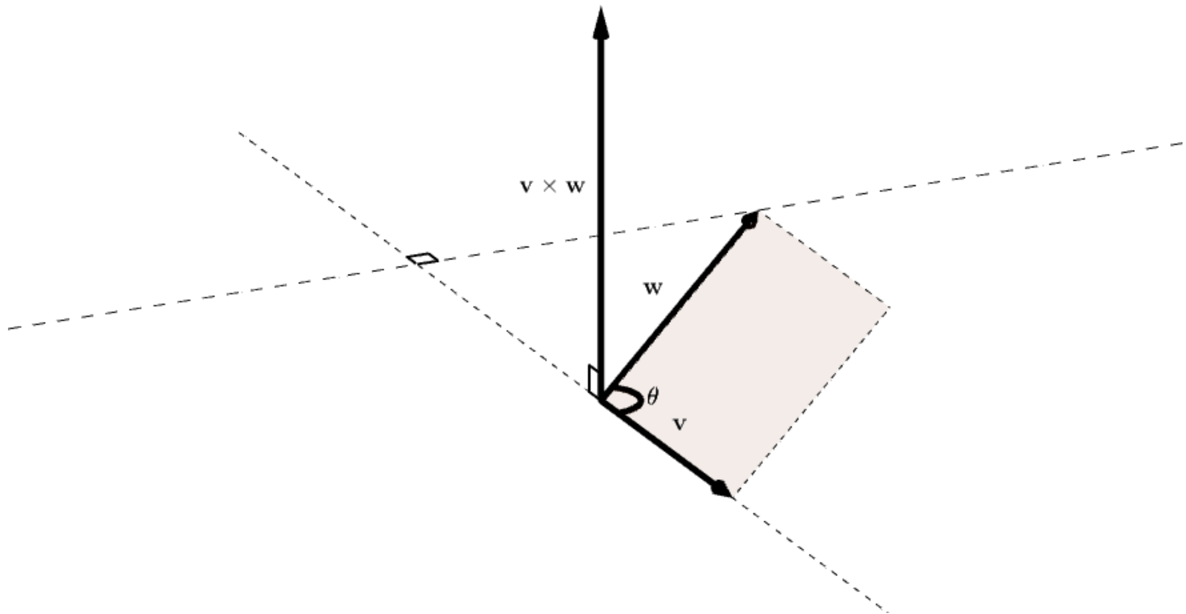
Definition 2.53. Now fix another vector $\mathbf{w} \in \mathbb{R}^3$. The **plane** containing P in the directions of $\mathbf{v}$ and $\mathbf{w}$ is the set of points

$$\{s\mathbf{v} + t\mathbf{w} + \mathbf{OP} \mid s, t \in \mathbb{R}\}$$

2.2.3 The Cross Product

Definition 2.54. The **cross product** of two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, which aren't scalar multiples of each other, is defined to be the unique vector, $\mathbf{v} \times \mathbf{w}$, such that it's orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, its direction is found using the right-hand rule, and its magnitude is equal in value to the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$. The next image proves

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta.$$



Define $\mathbf{v} \times \mathbf{0} := \mathbf{0}$ and $\mathbf{0} \times \mathbf{v} := \mathbf{0}$. If $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$, then $\mathbf{v} \times \mathbf{w} := \mathbf{0}$.

Proposition 2.55. If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, then

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

Exercise 2.56. Prove the previous proposition by following the outline in [Strang, 11.3]. First, let $\mathbf{x} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$.

(a) Justify. $\mathbf{x}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(b) Justify. $\|\mathbf{x}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2$.

(c) Informally describe why $\mathbf{x}$ and $\mathbf{v} \times \mathbf{w}$ are pointing in the same direction.

(d) Justify. $\mathbf{x} = \mathbf{v} \times \mathbf{w}$.

Proposition 2.57. (Properties of the cross product.) For any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, for any $a \in \mathbb{R}$, then

(a) (triple product) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$.

(b) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.

(c) $a(\mathbf{v} \times \mathbf{w}) = (a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w})$.

(d) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.

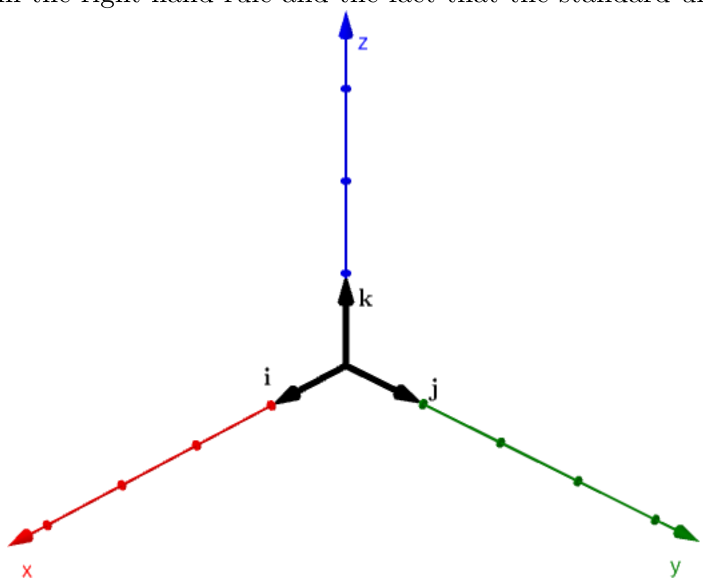
Proof. (b) and (c) follow directly from the definition. □

Exercise 2.58. Prove parts (a) and (d) of the previous proposition.

Example 2.59.

$$\mathbf{i} \times \mathbf{j} = \mathbf{k},$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j},$$

from the right-hand rule and the fact that the standard unit vectors pairwise span a unit square.



□

3 Vector valued functions and surface parametrizations

3.1 Vector valued functions (curve parametrizations)

Definition 3.1. Given real valued f, g, h defined on an open interval $(a, b) = \{t \in \mathbb{R} \mid a < t < b\}$, we call $\mathbf{r} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ a **vector valued function**. For any $t \in (a, b)$, $\mathbf{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$ is the position vector for the point $(f(t), g(t), h(t))$, so $\mathbf{r}$ is referred to as a **parametrization** of the curve $\mathcal{C} = \{(f(t), g(t), h(t)) \mid t \in (a, b)\}$ in 3-dimensional space, and the equations

$$x = f(t),$$

$$y = g(t),$$

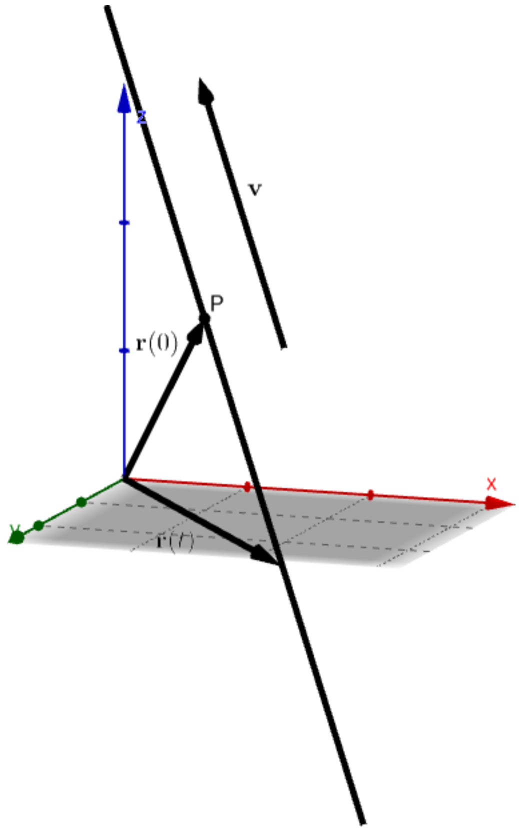
$$z = h(t),$$

are called **parametric equations** for $\mathcal{C}$.

Example 3.2. (a) For any $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ and any point $P = (P_1, P_2, P_3)$ in space, the vector valued function

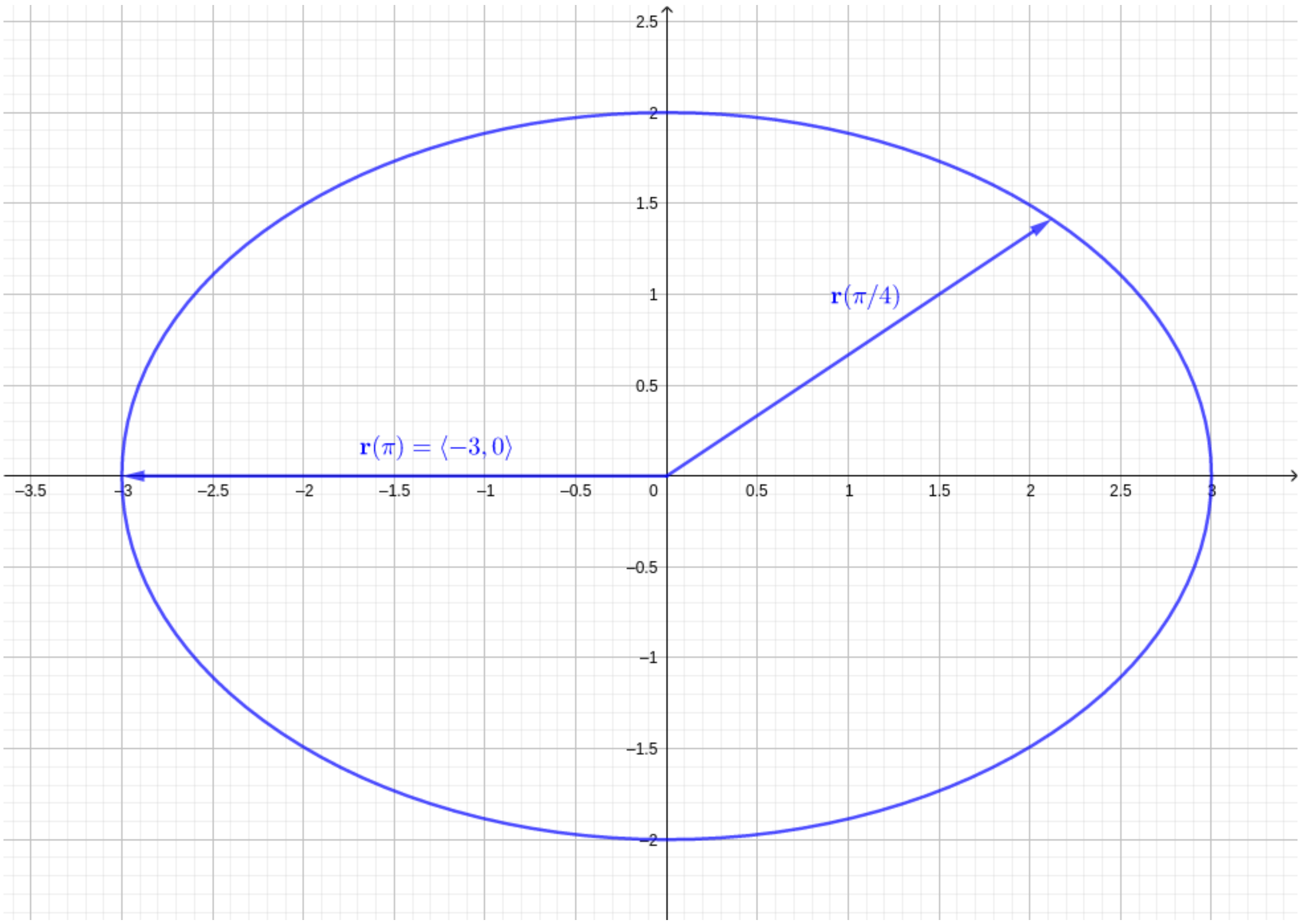
$$\mathbf{r}(t) = \begin{pmatrix} tv_1 + P_1 \\ tv_2 + P_2 \\ tv_3 + P_3 \end{pmatrix} = t\mathbf{v} + \mathbf{OP}$$

for $t \in \mathbb{R}$ parametrizes a line with direction $\mathbf{v}$ passing through P.



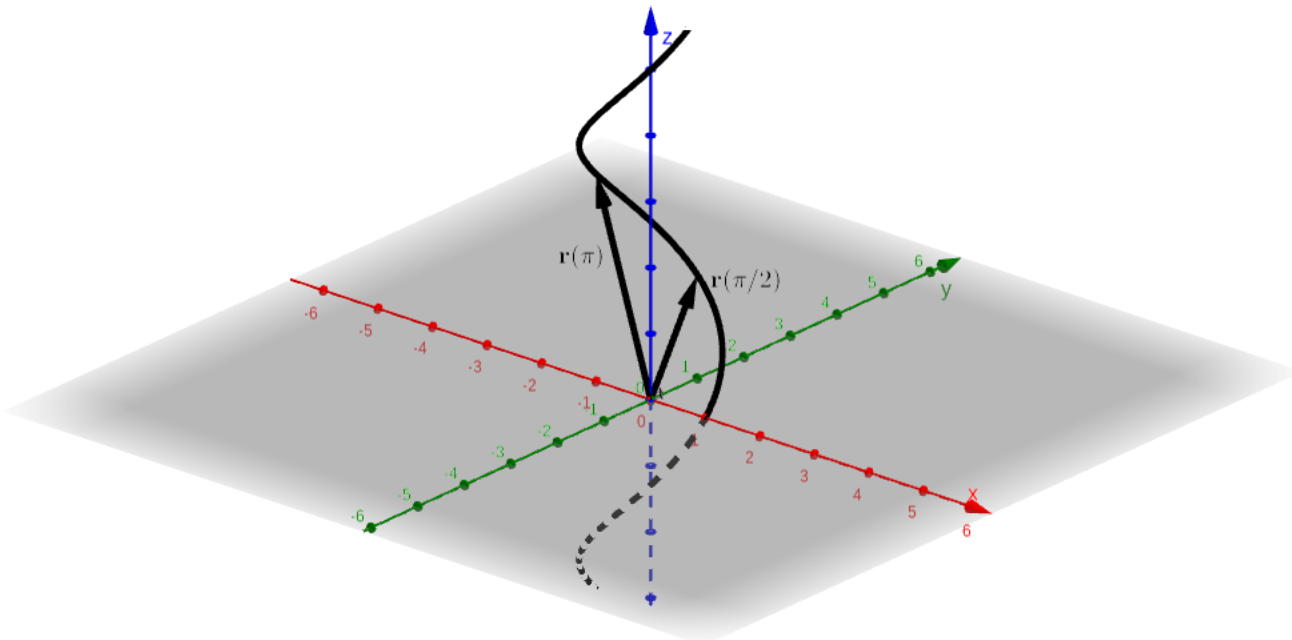
(b) $\mathbf{r}(t) = \begin{pmatrix} 3\cos(t) \\ 2\sin t \end{pmatrix}$ for $t \in [0, 2\pi]$ parametrizes an ellipse. Notice the planar curve described by the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is the ellipse parametrized by $\mathbf{r}$, since if $x(t) = 3\cos t$ and $y(t) = 2\sin t$, then

$$\frac{x^2(t)}{9} + \frac{y^2(t)}{4} = 1.$$



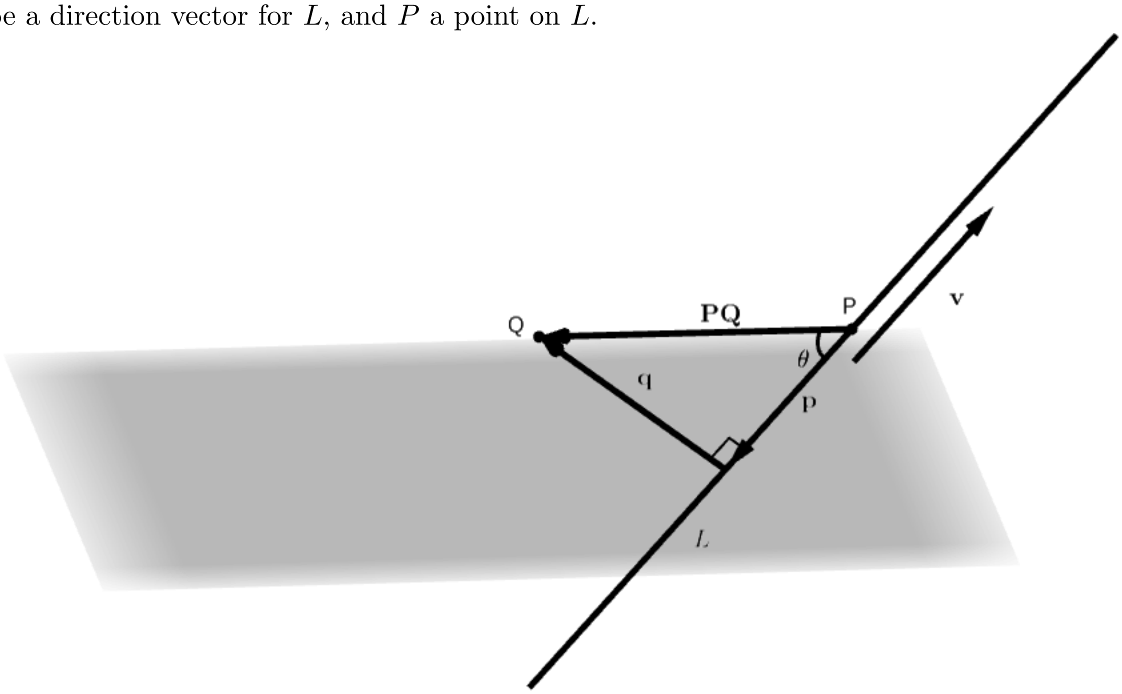
(c) To find the domain of a vector valued function, it's enough to find the domain of each component function. For example, the domain of $\mathbf{r}(t) = \begin{pmatrix} -\frac{1}{t-1} \\ \tan(\pi/2 - t) \\ \log t \end{pmatrix}$ is all $t \in \mathbb{R}$ such that $t > 0, t \neq 1$ and $t \neq n\pi$, for positive integers n.

(d) $\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$, $t \in \mathbb{R}$ parametrizes a **helix**.



□

Example 3.3. We can find the distance between a line, L, and a point, Q, in space as follows. Let $\mathbf{v}$ be a direction vector for L, and P a point on L.



Set $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{PQ} = \frac{\mathbf{PQ}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Then if $\mathbf{q} = \mathbf{PQ} - \mathbf{p}$, then $\|\mathbf{q}\|$ is the distance from L to Q, by the Pythagorean Theorem.

Now, by Proposition 2.20 and the definition of the dot product,

$$\begin{aligned}\|\mathbf{q}\|^2 = \|\mathbf{PQ} - \mathbf{p}\|^2 &= (\mathbf{PQ} - \mathbf{p}) \cdot (\mathbf{PQ} - \mathbf{p}) \\ &= \|\mathbf{PQ}\|^2 + \|\mathbf{p}\|^2 - 2\mathbf{PQ} \cdot \mathbf{p} \\ &= \|\mathbf{PQ}\|^2 + \frac{(\mathbf{PQ} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} - 2\frac{(\mathbf{PQ} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} \\ &= \frac{\|\mathbf{PQ}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2\theta)}{\|\mathbf{v}\|^2} \\ &= \frac{\|\mathbf{PQ} \times \mathbf{v}\|^2}{\|\mathbf{v}\|^2},\end{aligned}$$

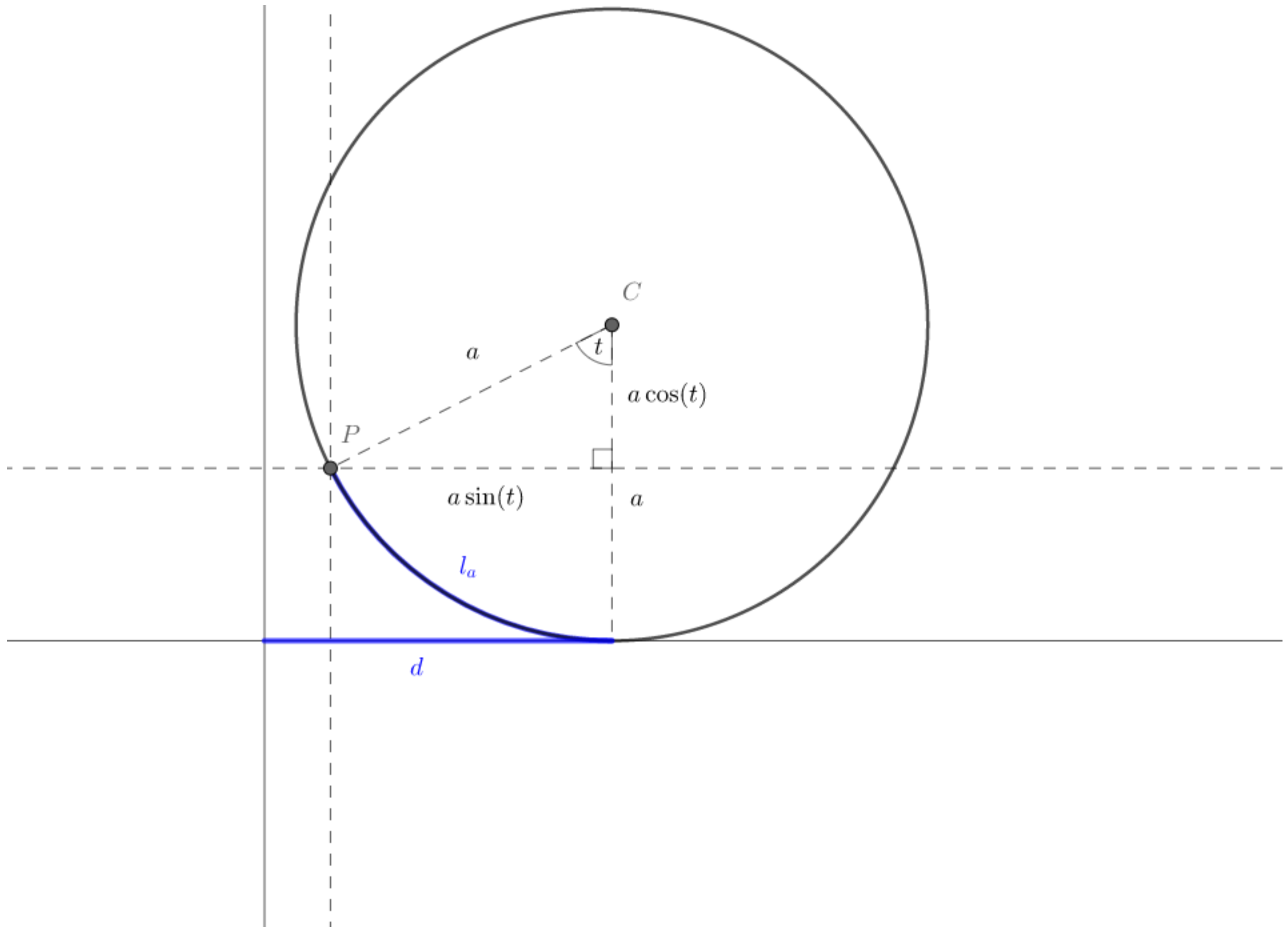
where the last line follows from the definition of the cross product.

That is, the distance between a line L and a point Q is

$$\operatorname{dist}(L, Q) = \|\mathbf{q}\| = \frac{\|\mathbf{PQ} \times \mathbf{v}\|}{\|\mathbf{v}\|},$$

where $\mathbf{q} = \mathbf{PQ} - \operatorname{proj}_{\mathbf{v}}\mathbf{PQ}$, P is an arbitrary point on L, and $\mathbf{v}$ is a direction vector for L. (Opinion: seeing $\operatorname{dist}(L, Q)$ as $\|\mathbf{q}\|$ is more geometric, while seeing $\operatorname{dist}(L, Q)$ as $\frac{\|\mathbf{PQ}\times\mathbf{v}\|}{\|\mathbf{v}\|}$ is more computationally friendly, at least by hand.) □

Example 3.4. A **cycloid** is a path traced out by a point on a circle moving in a straight line without slipping. Find a parametrization for this curve.



From the above image, $d = l_a$ by the no-slip condition. And $l_a = at$, which is the arc-length of the sector corresponding to the angle t the circle with radius a has rotated through. Then we see that $P = (at - a\sin t, a - a\cos t)$. So, a parametrization for a cycloid is

$$\mathbf{r}(t) = \begin{pmatrix} at - a\sin t \\ a - a\cos t \end{pmatrix}.$$

□

Definition 3.5. We say a vector valued function $\mathbf{r}$ approaches a vector $\mathbf{L}$ as t approaches a, written $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t\to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0$.

Theorem 3.6. If $\mathbf{r} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$, $\mathbf{L} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$, then $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$ if and only if $\lim_{t\to a} f(t) = L_1$, $\lim_{t\to a} g(t) = L_2$, and $\lim_{t\to a} h(t) = L_3$.

Proof. Both directions can be seen by a proof by contradiction. □

Definition 3.7. We say a vector valued function $\mathbf{r}$ is **continuous** at $t = a$ provided

1. $\mathbf{r}(a)$ exists.
2. $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

We say $\mathbf{r}$ is continuous on an open interval (a, b) if $\mathbf{r}$ is continuous at every point $t \in (a, b)$. Similarly with closed and half-open intervals, with the appropriate conditions on the boundary.

Example 3.8. $\mathbf{r}(t) = \begin{pmatrix} \sin t \cos t \\ e^{-t} \\ t^2 + 1 \end{pmatrix}$ is continuous on $\mathbb{R}$ by Theorem 3.6 since its components are continuous on $\mathbb{R}$. □

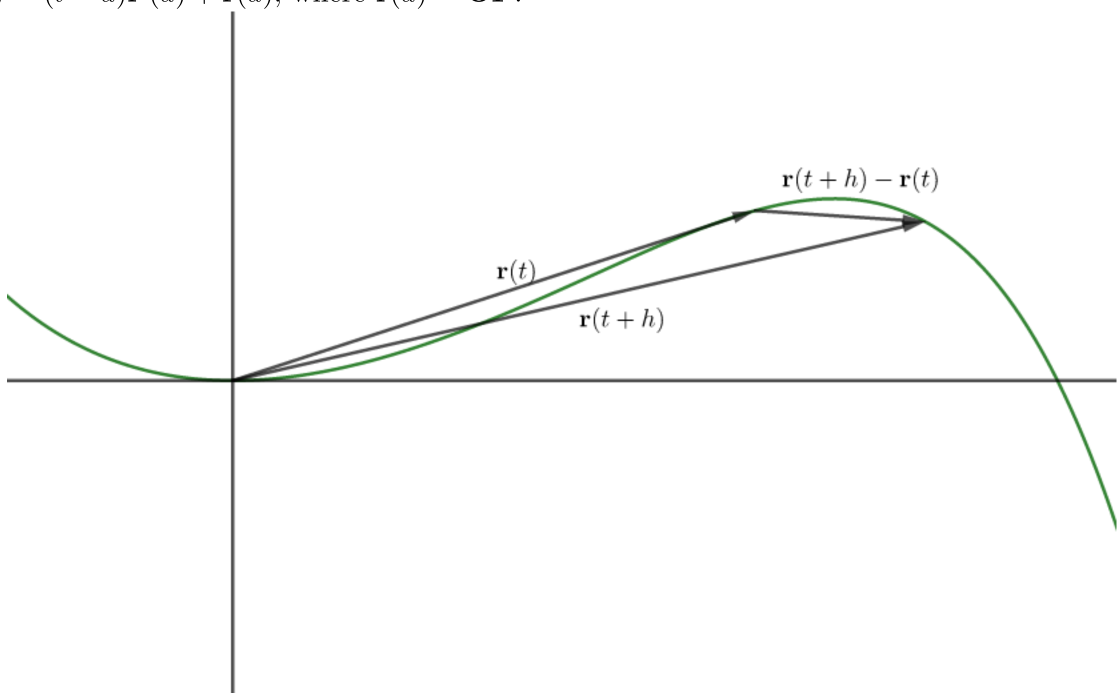
Definition 3.9. We say a vector valued function $\mathbf{r}$ is **differentiable** at t if $\lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ exists. If this is the case, we denote

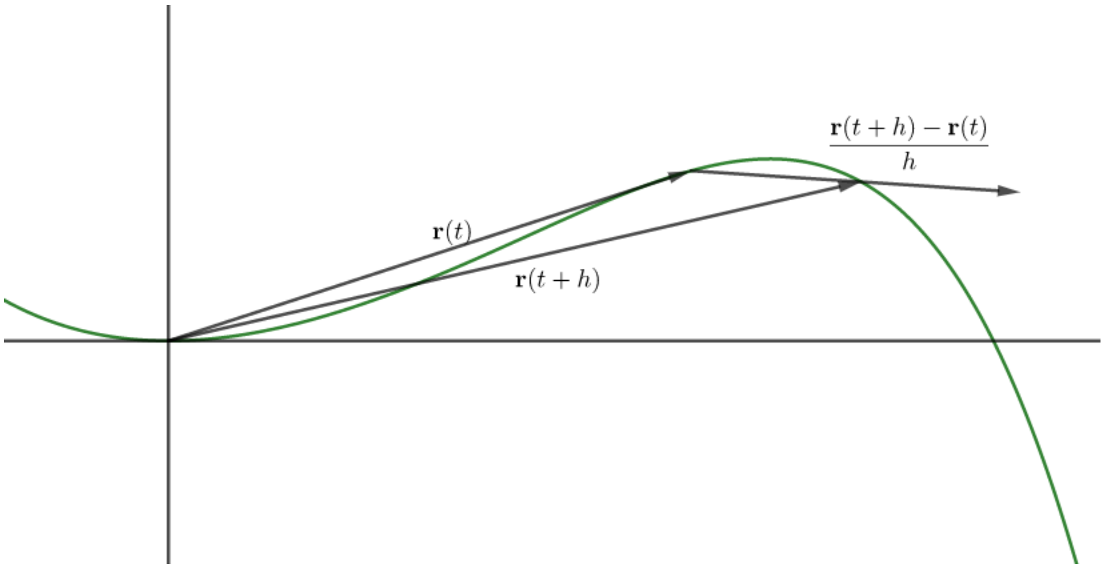
$$\mathbf{r}'(t) := \frac{d}{dt}\mathbf{r}(t) := \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

called the **tangent vector** to $\mathbf{r}$ at t.

We say $\mathbf{r}$ is **differentiable** on (a, b) if $\mathbf{r}$ is differentiable for every $t \in (a, b)$. Similarly for closed and half-open intervals, with the appropriate conditions on the boundary. See [**OS** Section 3.2 page 268].

The **tangent line** to a curve $\mathcal{C}$ at a point P parametrized by $\mathbf{r}$ is the line parametrized by $\mathbf{l}(t) = (t - a)\mathbf{r}'(a) + \mathbf{r}(a)$, where $\mathbf{r}(a) = \mathbf{OP}$.





Theorem 3.10. A vector valued function $\mathbf{r} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ is differentiable at t if and only if f, g, h are at t. In this case, $\mathbf{r}'(t) = \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix}$.

Proof. This follows from Theorem 3.6 and Proposition 2.39 (for space vectors). □

Example 3.11. If $\mathbf{r}(t) = \begin{pmatrix} \log(t-1) \\ e^{2t} + t \\ \tan(t) \end{pmatrix}$, then $\mathbf{r}'(t) = \begin{pmatrix} \frac{1}{t-1} \\ 2e^{2t} + 1 \\ \sec^2(t) \end{pmatrix}$. □

Theorem 3.12. Properties of the derivative. For any real-valued function, f, vector valued functions $\mathbf{c}, \mathbf{r}$, we have

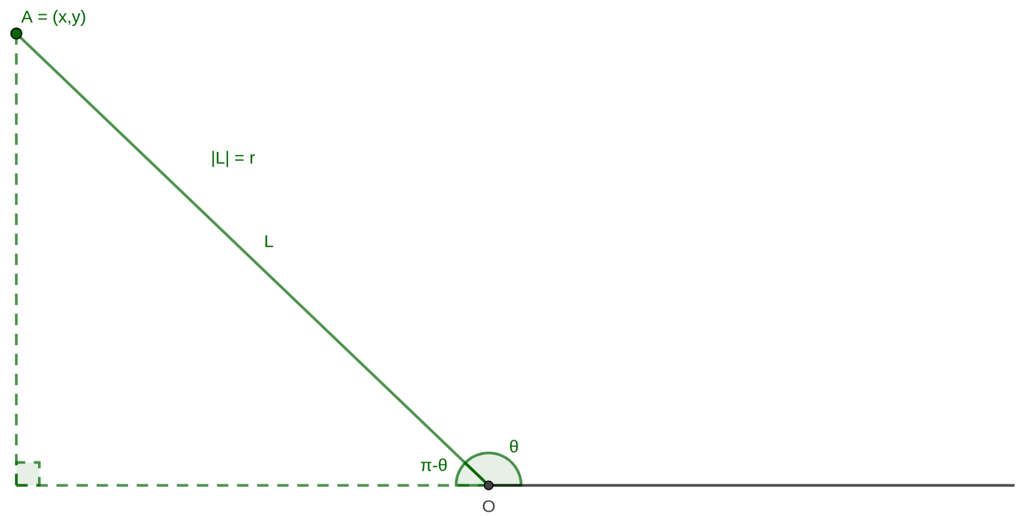
i. $\frac{d}{dt}(\mathbf{c} \pm \mathbf{r})(t) = \mathbf{c}'(t) \pm \mathbf{r}'(t)$
ii. $\frac{d}{dt}(\mathbf{c} \cdot \mathbf{r})(t) = \mathbf{c}'(t) \cdot \mathbf{r}(t) + \mathbf{c}(t) \cdot \mathbf{r}'(t)$
iii. $\frac{d}{dt}(\mathbf{c} \times \mathbf{r})(t) = \mathbf{c}'(t) \times \mathbf{r}(t) + \mathbf{c}(t) \times \mathbf{r}'(t)$
iv. $\frac{d}{dt}(f\mathbf{r})(t) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
v. $\frac{d}{dt}\mathbf{r}(f(t)) = f'(t)\mathbf{r}'(f(t))$

Proof. The proofs of these properties are standard following from Theorem 3.10. The first property is one of linearity. The middle three are product rules, while the last is the chain rule for vector valued functions. □

Remark 3.13. The (in)definite integral(s) of a vector valued function are described in the same way (component-wise). See [**OS** pages 274-276].

4 polar coordinates

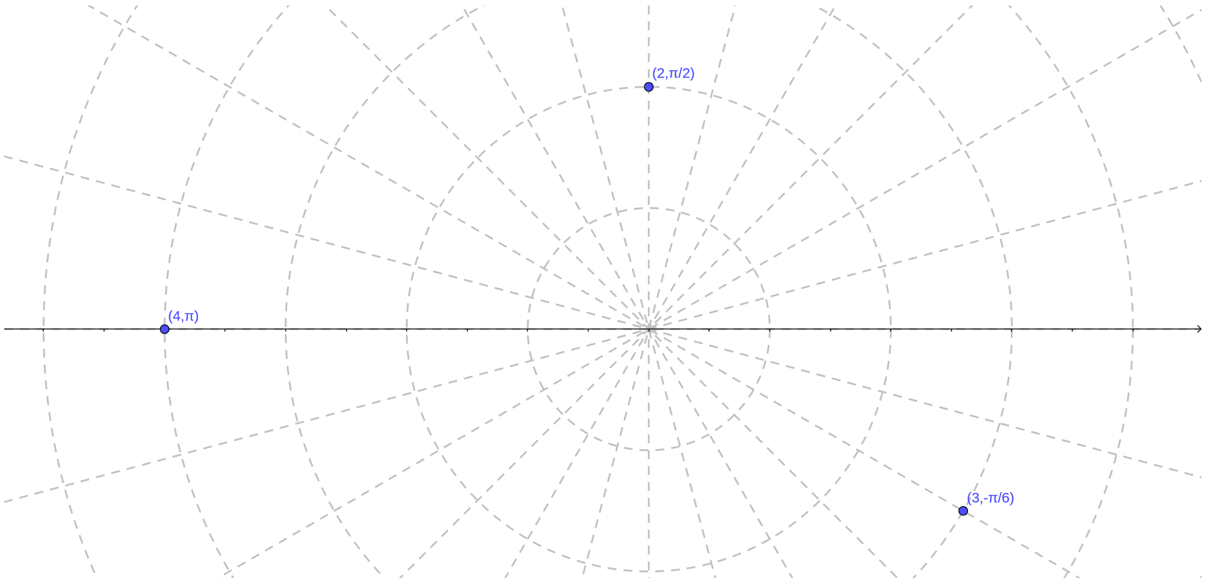
Exercise 4.1. Draw a horizontal ray starting at a point, O, in a plane. Label a point, A, not on this ray. Draw the line segment, L, from O to A. Label the length, r, of L. Label the angle, θ, from the ray to L in the counter-clockwise direction. Now draw a rectangular coordinate system where the nonnegative x-axis is the horizontal ray and the origin is the point O. If $A = (x, y)$ in this coordinate system, write x and y in terms of r and θ.



Specifically, this is the map relating polar coordinates to rectangular coordinates. You found $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Polar coordinates are another way of describing points in the plane. For each point, we still need an ordered pair of real numbers, (r, θ), but instead of each being the signed distance from the origin along perpendicular axes, r is the distance between the point and the origin, θ is the angle between a fixed ray and the ray from the origin to the point, measured either counterclockwise (if $\theta > 0$) or clockwise (if $\theta < 0$), in between $-\pi$ and π. The fixed ray is called the **polar axis**, and is usually drawn where the positive x-axis is drawn. Instead of a rectangular grid, we draw concentric circles emanating from the origin. These circles have constant r. The constant θ curves are rays emanating from the origin.

We plot a few points in the plane using polar coordinates. $r > 0$ and $-\pi < \theta \leq \pi$.



When $r > 0$ and $-\pi < \theta \leq \pi$, each (r, θ) in polar coordinates corresponds to exactly one point in the plane except the origin, which we denote by $(0, 0)$. Conversely, every point in the plane other than the origin corresponds to exactly one (r, θ) with $r > 0$ and $-\pi < \theta \leq \pi$. Importantly, we can convert polar coordinates to rectangular coordinates. We've already done this. If (x, y) in rectangular coordinates describes the same point (r, θ) in polar coordinates, then

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta).$$

Example 4.2. For example, in the previous image,

$(2, \pi/2)$ in polar describes the same point as $(0, 2)$ in rectangular.

$(3, -\pi/6)$ in polar describes the same point as $(3\sqrt{3}/2, -3/2)$.

$(4, \pi)$ in polar is $(-4, 0)$ in rectangular.

Exercise 4.3. Given the following points in polar coordinates, convert to rectangular coordinates:

(a) $(4, \frac{2\pi}{3})$

(b) $(3, -\frac{\pi}{4})$.

(c) $(\sqrt{3}, \frac{\pi}{12}$

We may also convert from rectangular to polar. If (x, y) describes a point in rectangular coordinates, then from the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$r^2 = x^2 + y^2.$$

And

$$\tan(\theta) = \frac{y}{x}.$$

Then $r = \sqrt{x^2 + y^2}$ since $r > 0$. Some care is needed when solving for θ in this last equation though. Remember the range of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means $\theta = \arctan\left(\frac{y}{x}\right)$ only when $x > 0$. Otherwise, $|\arctan\left(\frac{y}{x}\right)|$ is the reference angle of the angle describing the point (x, y) in polar coordinates. Thus, if $x < 0$ and $y \geq 0$, then

$$\theta = \arctan\left(\frac{y}{x}\right) + \pi.$$

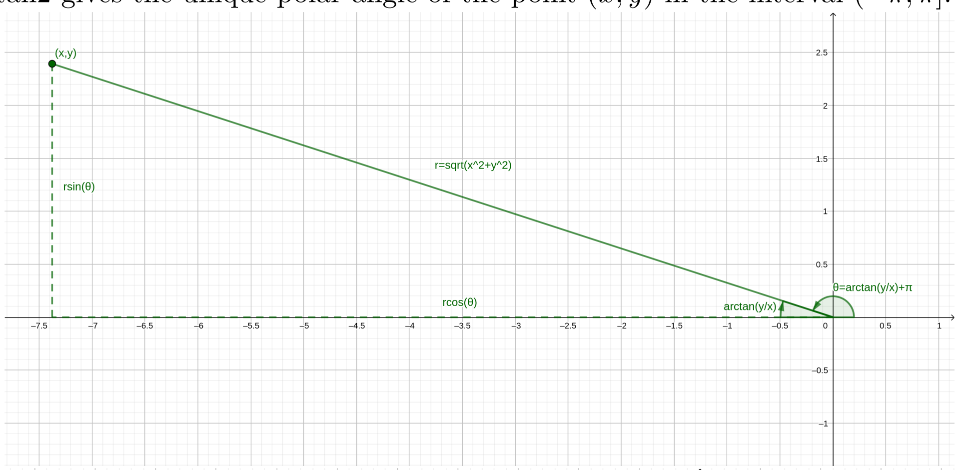
If $x < 0$ and $y < 0$, then

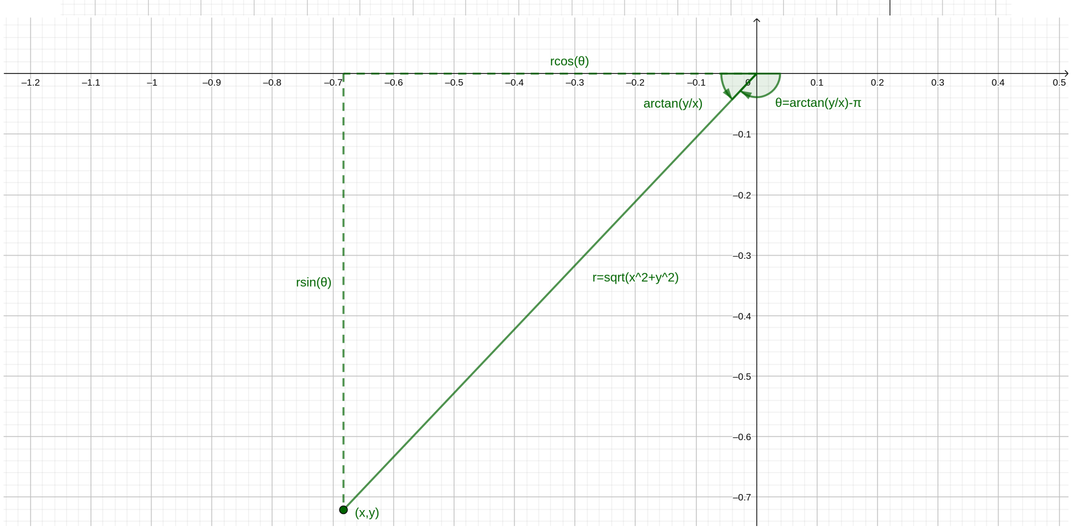
$$\theta = \arctan\left(\frac{y}{x}\right) - \pi.$$

We gather these data in a single function, called arctan2, since there are two inputs.

$$\arctan 2(x, y) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0. \end{cases}$$

Notice this function is not defined at the origin, when $x = y = 0$, but is the only point excluded. Otherwise, arctan2 gives the unique polar angle of the point (x, y) in the interval $(-\pi, \pi]$.





Example 4.4. For example, let's convert $(0,-2)$, $(-1,\sqrt{3})$, and $(-\sqrt{2}/2,-\sqrt{2}/2)$ to polar coordinates.

$$(0,-2)$$

in rectangular becomes

$$(2,-\pi/2)$$

in polar.

$$(-1,\sqrt{3})$$

in rectangular becomes

$$(\sqrt{1+3}, \arctan(-\sqrt{3})+\pi) = \left(2, \frac{2\pi}{3}\right)$$

in polar.

$$(-\sqrt{2}/2,-\sqrt{2}/2)$$

in rectangular becomes

$$\left(\sqrt{\frac{1}{2}+\frac{1}{2}}, \arctan(1)-\pi\right) = \left(1, -\frac{3\pi}{4}\right)$$

in polar.

Exercise 4.5. Given the following points in rectangular coordinates, convert to polar coordinates:

1. $(-1,1)$
2. $(0,4)$
3. $(\sqrt{6},\sqrt{2})$.

4.1 graphs in polar coordinates

Example 4.6. Consider the following equations in rectangular coordinates.

(a) If $x^2+y^2=1$, then in polar coordinates, $r=1$, a simple equation whose solution set is the unit circle.

(b) If $x^2+y^2=6y$, then $r^2=6r\sin(\theta)$, which, if $r>0$, then $r=6\sin(\theta)$. This is a circle which intersects the origin, has center $(0,3)$ with radius 3.

(c) Similarly, we can write any line $y=mx+b$ in polar form by writing $x=r\cos(\theta)$ and $y=r\sin(\theta)$ and replacing

$$r\sin(\theta)=mr\cos(\theta)+b.$$

If we wish to write r as a function of θ, we can write

$$r=\frac{b}{\sin(\theta)-m\cos(\theta)}.$$

We notice, if $b \neq 0$, as $\tan(\theta)$ approaches m, r approaches $\pm\infty$. What is an appropriate domain of this function? Certainly, $\tan(\theta) \neq m$. We choose the union

$$\left[-\frac{\pi}{2}, \arctan(m)\right) \cup \left(\arctan(m), \frac{\pi}{2}\right].$$

If $b = 0$, the line passing through the origin with slope m is given by

$$\theta = \arctan(m).$$

Example 4.7. Conversely, we can convert from polar to rectangular.

(a) If

$$r = 5\tan(\theta),$$

then

$$r^2\cos(\theta) = 5r\sin(\theta)$$

implies

$$x\sqrt{x^2+y^2} = 5y.$$

Hence,

$$x^2(x^2+y^2) = 25y^2.$$

Hence,

$$y^2 = \frac{x^4}{25-x^2}.$$

So, this is two copies of a radical function with a rational radicand.

(b) We rewrite

$$r = \cos(2\theta)$$

as

$$r = \cos^2(\theta) - \sin^2(\theta).$$

Hence,

$$r^3 = r^2\cos^2(\theta) - r^2\sin^2(\theta),$$

so that

$$(x^2+y^2)^{3/2} = x^2 - y^2.$$

Hence,

$$(x^2+y^2)^3 = (x^2-y^2)^2.$$

Definition 4.8. We consider what it means for a point in polar coordinates to have a nonpositive r-coordinate or a θ-coordinate outside of the interval $(-\pi, \pi]$.

If either $\theta > \pi$ or $\theta \leq -\pi$, then there is a unique $\theta_0 \in (-\pi, \pi]$ and unique integer k for which $\theta = \theta_0 + 2\pi k$. Then if $r > 0$, map (r, θ) to (r, θ_0).

We map $(0, \theta)$ to the origin, no matter θ. If $r < 0$, we map (r, θ) to $(-r, \theta + \pi)$. If $\theta + \pi$ is outside of $(-\pi, \pi]$, then we use the same procedure as above to identify it with a point in standard polar coordinates, with its θ-coordinate in $(-\pi, \pi]$.

Exercise 4.9. Graph the following curves.

(a) $r = \theta$.

(b) $\theta = \ln(r)$.

(c) $r = \theta^2$.

(d) $r = \tan(\theta)$.

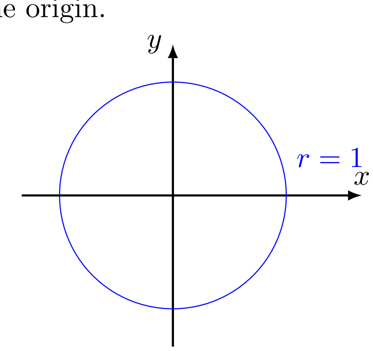
We will consider functions of the form $r = f(\theta)$ in polar coordinates. That is, we ask, what curves do we obtain when r depends on θ in a functional way? This is akin to considering functions of the form $y = f(x)$ in rectangular coordinates.

4.1.1 Circles and limaçons

We now describe a family of curves. $r = a \pm b\cos(\theta)$ and $r = a \pm b\sin(\theta)$ with $a \geq 0$, and $b \geq 0$.

Example 4.10. Case 1: $b = 0$

This is the simplest curve we can ask for, the case $a = 1$ considered in a previous example. If $r = a$, then r does not depend on θ. If $a > 0$ and $r = a$, then all of the points (r, θ) which satisfy this equation lie on a circle of radius a centered at the origin. For further verification, recall $r = \sqrt{x^2 + y^2}$ in the xy-plane. These are circles of constant r, just as, for example, the equation $x = a$ in rectangular coordinates is the vertical line passing through $(a, 0)$. If $a = 0$, $r = 0$ is just the origin.



Example 4.11. Case 2: $a = 0$**,** $b > 0$

If $r = \pm b\cos(\theta)$, or $r = \pm b\sin(\theta)$, then we claim the graph of this function in polar coordinates is another circle, the case $b = 6$ considered in a previous example. Suppose $r = b\cos(\theta)$. The maximum of r occurs when $\theta = 0$. $r = b$ in this case. r is a minimum when $\theta = \pi$, and $r = -b$. But since $b > 0$, $(-b, \pi)$ is the same as $(b, 0)$, where r is a maximum. So, the minimum is instead when $r = 0$, which occurs when $\theta = \frac{\pi}{2}$. Hence, if this is a circle, then its diameter is b.

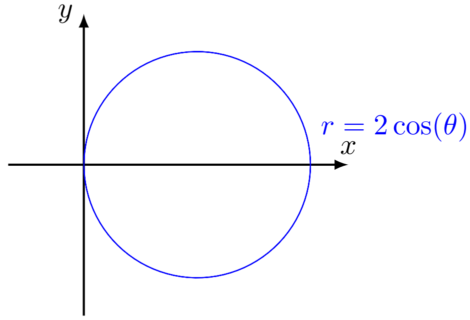
To notice this is a circle, we multiply $r^2 = br\cos(\theta)$, and write in rectangular coordinates

$$x^2 - bx + y^2 = 0.$$

We complete the square to obtain

$$\left(x - \frac{b}{2}\right)^2 + y^2 = \frac{b^2}{4}.$$

So, this is a circle with radius $b/2$ and center $(b/2, 0)$ in rectangular coordinates.



Example 4.12. Case 3: $a > 0$**,** $b > 0$

These curves are called limaçons. We have the following subcases:

If $\frac{a}{b} \geq 2$, these are convex limaçons.

If $1 < \frac{a}{b} < 2$, these are dimpled limaçons.

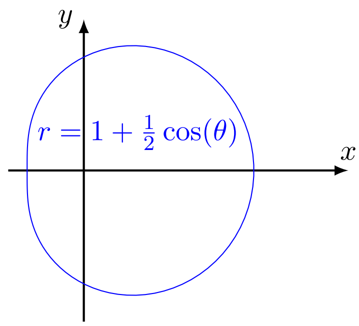
If $\frac{a}{b} = 1$, these are cardioids.

If $\frac{a}{b} < 1$, these are inner-loop limaçons.

Example 4.13. Suppose $r = 1 + \frac{1}{2}\cos(\theta)$, whose graph is a convex limaçon. What we mean by convex is that any two points in the interior of this curve can be joined by a straight line which lies entirely in the interior of the curve.

θ	$r = 1 + \frac{1}{2}\cos(\theta)$
0	$\frac{3}{2}$
$\frac{\pi}{6}$	$1 + \frac{\sqrt{3}}{4}$
$\frac{\pi}{4}$	$1 + \frac{\sqrt{2}}{4}$
$\frac{\pi}{3}$	$\frac{5}{4}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{4}$
$\frac{3\pi}{4}$	$1 - \frac{\sqrt{2}}{4}$
$\frac{5\pi}{6}$	$1 - \frac{\sqrt{3}}{4}$
π	$\frac{1}{2}$

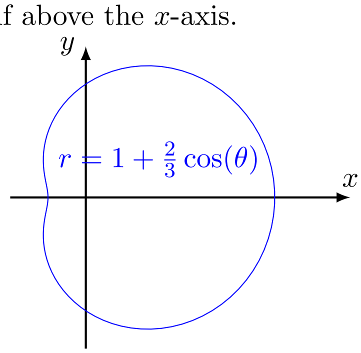
This curve is certainly symmetric about the rays $\theta = 0$ and $\theta = \pi$, so we only need to consider the half above the x-axis.



Example 4.14. Suppose $r = 1 + \frac{2}{3}\cos(\theta)$, whose graph is called a dimpled limaçon.

θ	$r = 1 + \frac{2}{3}\cos(\theta)$
0	$\frac{5}{3}$
$\frac{\pi}{6}$	$1 + \frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$1 + \frac{\sqrt{2}}{3}$
$\frac{\pi}{3}$	$\frac{4}{3}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{2}{3}$
$\frac{3\pi}{4}$	$1 - \frac{\sqrt{2}}{3}$
$\frac{5\pi}{6}$	$1 - \frac{\sqrt{3}}{3}$
π	$\frac{1}{3}$

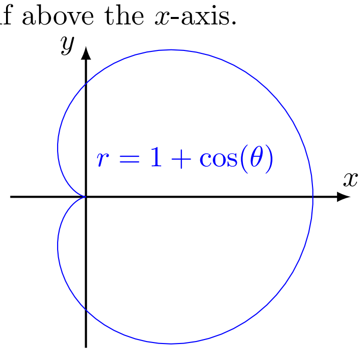
This curve is certainly symmetric about the rays $\theta = 0$ and $\theta = \pi$, so we only need to consider the half above the x-axis.



Example 4.15. Suppose $r = 1 + \cos(\theta)$, whose graph is called a cardioid, since it resembles a heart.

θ	$r = 1 + \cos(\theta)$
0	2
$\frac{\pi}{6}$	$1 + \frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$1 + \frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{3}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{1}{2}$
$\frac{3\pi}{4}$	$1 - \frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$1 - \frac{\sqrt{3}}{2}$
π	0

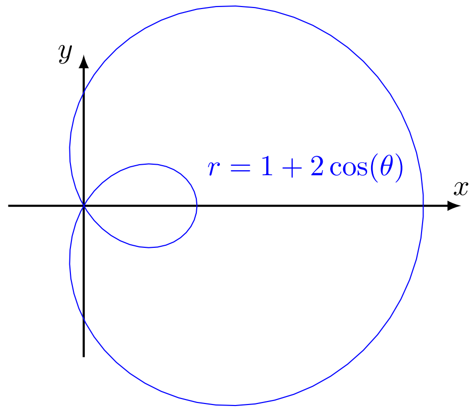
This curve is certainly symmetric about the rays $\theta = 0$ and $\theta = \pi$, so we only need to consider the half above the x-axis.



Example 4.16. Suppose $r = 1 + 2\cos(\theta)$, whose graph is called an inner loop limaçon.

θ	$r = 1 + 2\cos(\theta)$
0	3
$\frac{\pi}{6}$	$1 + \sqrt{3}$
$\frac{\pi}{4}$	$1 + \sqrt{2}$
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	0
$\frac{3\pi}{4}$	$1 - \sqrt{2}$
$\frac{5\pi}{6}$	$1 - \sqrt{3}$
π	-1

Notice the last three rows correspond to the points $\left(\sqrt{2} - 1, -\frac{\pi}{4}\right)$, $\left(\sqrt{3} - 1, -\frac{\pi}{6}\right)$, $(1, 0)$ respectively.



It's helpful to imagine this family of curves as a whole. For definiteness, consider $r = a + b\cos(\theta)$ with $a, b > 0$. Notice they all agree at $\theta = \frac{\pi}{2}$. Notice as b increases from zero, the limaçon starts to flatten near $\theta = \pi$. Then when $a/b = 2$, the limaçon starts to dimple, or bend toward the origin at $\theta = \pi$. When $a/b = 1$, the curve intersects the origin in a pinch at $\theta = \pi$, producing a cardioid. When $a/b < 1$, the pinch folds in on itself to create another loop.

Another way to consider these curves is as parametrized curves in the plane. Remember in rectangular coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. If, for example, $r = a + b\cos(\theta)$, then the curve given by

$$c(\theta) = ((a + b\cos(\theta))\cos(\theta), (a + b\cos(\theta))\sin(\theta))$$

for all θ describes the graph of $r = a + b\cos(\theta)$.

Notice we've described curves of the form $r = a + b\cos(\theta)$. The curves given by $r = a - b\cos(\theta)$ or $r = a \pm b\sin(\theta)$ are simply rotations of one of the others by $\frac{\pi}{2}$ in the counterclockwise direction.

Exercise 4.17. Graph the following.

(a) $r = 2 + \sin(\theta)$.

(b) $r = 3 - 2\sin(\theta)$.

(c) $r = 2 - \cos(\theta)$.

(d) $r = 1 + 2\sin(\theta)$.

4.1.2 Lemniscates and roses

We now consider more complicated graphs. The first are Lemniscates. These are curves for which $a \neq 0$,

$$r^2 = \pm a^2\cos(2\theta)$$

and

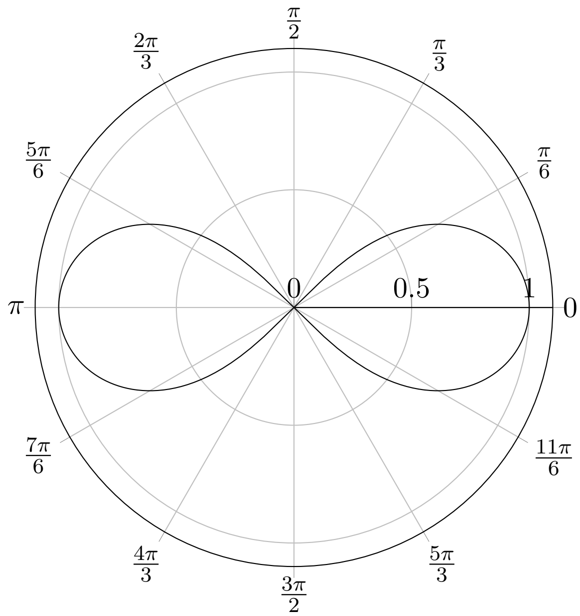
$$r^2 = \pm a^2\sin(2\theta).$$

Each one of these curves is a rotation of $\frac{\pi}{4}$ by one of the others. So, again, we'll focus on the subfamily of curves

$$r^2 = a^2\cos(2\theta).$$

Example 4.18. Consider the curve $r^2 = \cos(2\theta)$. The domain of this function is restricted. θ can't be such that $\cos(2\theta) < 0$.

This curve is symmetric about the x-axis, sometimes referred to as the polar axis. So, we can trace out this curve. Remember the second column is r^2 for a given θ.



Exercise 4.19. What is the domain of $r^2 = \cos(2\theta)$, as a subset of $(-\pi, \pi]$?

The last family of curves we will investigate are those given by

$$r = a\sin(n\theta)$$

or

$$r = a\cos(n\theta)$$

with $a \neq 0$ and n an integer. This curve is called a rose. $r = a\sin(n\theta)$ is a rotation of $r = a\cos(n\theta)$ by an angle of $\frac{\pi}{2n}$. Also, $r = -a\cos(n\theta)$ is a rotation of $r = a\cos(\theta)$ by $\frac{\pi}{n}$. So, again, we consider curves of the form

$$r = a\cos(n\theta)$$

with $a > 0$. Now, since cos is even, we need only consider $n > 0$. If $n = 1$, then this is a circle we've already considered. So, we consider $n > 1$.

Example 4.20. Let's consider

$$r = \cos(2\theta).$$

θ	$r = \cos(2\theta)$
0	1
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	$-\frac{1}{2}$
$\frac{\pi}{2}$	-1
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	0
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	1

Again,

$$\left(-\frac{1}{2}, \frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{4\pi}{3}\right) = \left(\frac{1}{2}, -\frac{2\pi}{3}\right)$$

$$\left(-1, \frac{\pi}{2}\right) = \left(1, \frac{3\pi}{2}\right) = \left(1, -\frac{\pi}{2}\right)$$

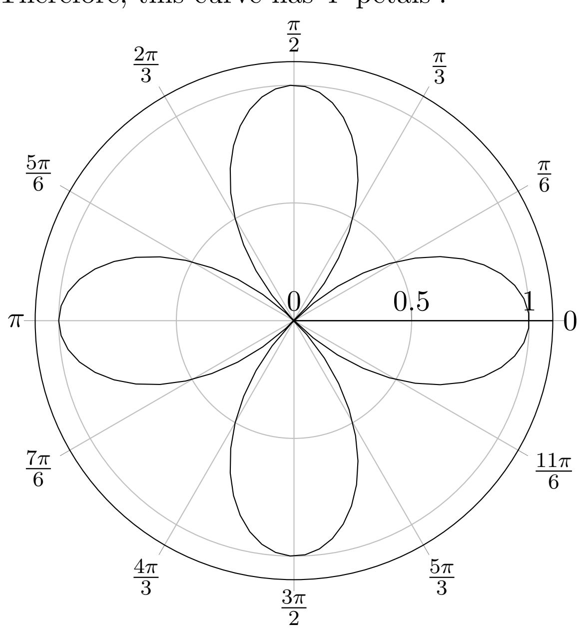
and

$$\left(-\frac{1}{2}, \frac{2\pi}{3}\right) = \left(\frac{1}{2}, \frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\pi}{3}\right)$$

and this curve is symmetric about the x-axis. This means if $(r\cos(\theta), r\sin(\theta))$ is on the graph, then so is $(r\cos(\theta), -r\sin(\theta))$. This is true for if $r = f(\cos(\theta))$ for any f, a nonnegative function with arguments in $[-1, 1]$. This is because

$$(f(\cos(\theta))\cos(\theta), -f(\cos(\theta))\sin(\theta))$$
$$= (f(\cos(-\theta))\cos(-\theta), f(\cos(-\theta))\sin(-\theta)).$$

Therefore, this curve has 4 'petals'.



Example 4.21. Let's consider

$$r = \cos(3\theta).$$

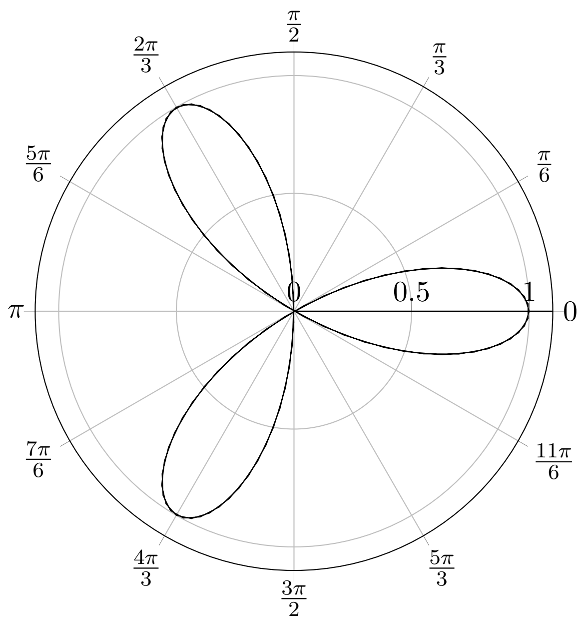
θ	$r = \cos(3\theta)$
0	1
$\frac{\pi}{6}$	0
$\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	-1
$\frac{\pi}{2}$	0

Again,

$$\left(-\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, -\frac{3\pi}{4}\right).$$

$$\left(-1, \frac{\pi}{3}\right) = \left(1, \frac{4\pi}{3}\right) = \left(1, -\frac{2\pi}{3}\right).$$

This curve is symmetric about the x-axis. Therefore, this curve has three 'petals'.



In general, if $r = a\cos(n\theta)$, then this curve has n petals if n is odd, and $2n$ petals if n is even.

Exercise 4.22. Graph the following.

(a) $r = \cos(5\theta)$.

(b) $r = \sin(2\theta)$.

(c) $r = 1 - 2\sin(3\theta)$. [Strang, pg 355]

4.2 Length and Area

4.2.1 Tangent Vectors

Since a polar equation corresponds to a parametrized curve, its tangent vectors are computed thusly: if $r = f(\theta)$,

$$\mathbf{r}'(\theta) = \begin{pmatrix} f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) \\ f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \end{pmatrix}$$

Exercise 4.23.

(a) Find the polar coordinates of where the curves $x = 1 - y^2$ and $r = 1 + 2\cos(\theta)$ are parallel.

(b) Find the points where the speed of the curve $r = \cos(\theta) + \sin(\theta)$ is at a maximum.

4.2.2 Length

Recall the length of a parameterized curve $\mathbf{r}$ defined on $[a, b]$ is $\int_a^b ds = \int_a^b |r'(t)|dt$.

Exercise 4.24. The length of a polar curve corresponding to $r = f(\theta)$ if $\theta \in [a, b]$ is

$$\int_a^b ds = \int_a^b \sqrt{f'(\theta) + f(\theta)} d\theta.$$

Exercise 4.25. Find the length of the following curves. $a \in \mathbb{R}$.

(a) $r = a$ if $\theta \in [-\pi, \pi]$.

(b) $r = a\sin(\theta)$ if $\theta \in [0, \pi]$.

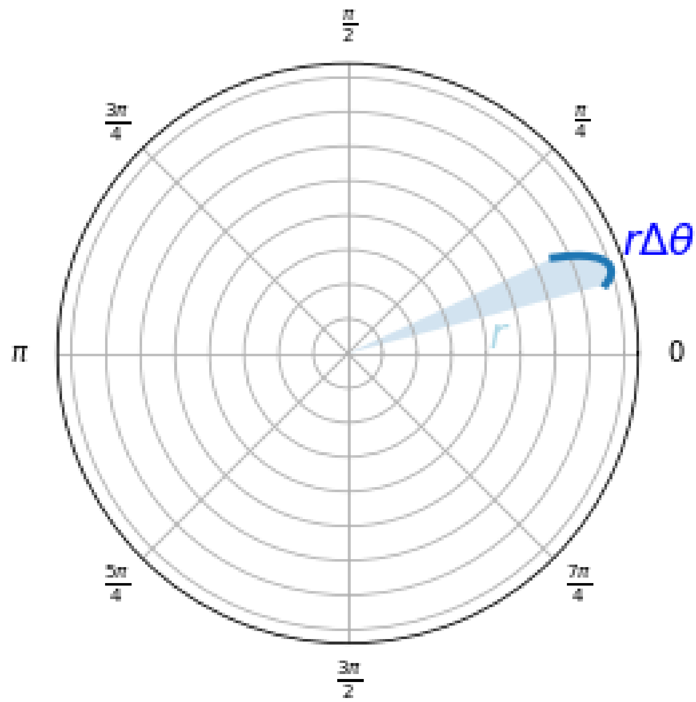
(c) $r = e^{-\theta}$ if $\theta \in [-\pi, \pi]$. [Strang, pg 359]

(d) $r = \theta^2$ if $\theta \in [-\pi, \pi]$. [Strang, pg 360]

Exercise 4.26. Parameterize the curve $r = \sin(\theta)$ such that is has unit speed.

4.2.3 Area

The area of a small piece of a polar graph, when $\Delta\theta$ (the difference in θ) is small, is approximately $\frac{\Delta\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\Delta\theta$, and this approximation gets better the smaller $\Delta\theta$ is.



Hence, if we partition the domain of $r = f(\theta)$, add up these small wedges and take the limit as $\Delta\theta \to 0$, then the area inside $r = f(\theta)$ is

$$\int_a^b \frac{1}{2}(f(\theta))^2 d\theta$$

if a and b are appropriate bounds, tracing the graph of $f(\theta)$ exactly once.

Note this is consistent with what is covered in the next class, integration by substitution: If φ is a continuously differentiable injective map from $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$ such that $\det(D\,\varphi(x)) \neq 0$ for all $x \in U$, then if f is a continuous function which vanishes outside of a compact set, then

$$\int_{\varphi(U)} f = \int_U f \circ \varphi \cdot |\det(D\,\varphi)|$$

In the polar case, $\varphi(r, \theta) = (r\cos(\theta), r\sin(\theta))$ and so

$$\det((D\,\varphi)(r, \theta)) = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r.$$

And the area of a region in the plane V is defined as the integral of the function 1 over V. If $U = \varphi^{-1}(V) = \{(r, \theta) \mid 0 \leq r \leq f(\theta), \theta \in [a, b]\}$, and f is continuous, then

$$\text{Area}(V) = \int_V = \int_{\varphi(U)} = \int_U r = \int_a^b \int_0^{f(\theta)} r\,dr\,d\theta = \int_a^b \frac{1}{2}(f(\theta))^2 d\theta.$$

Example 4.27. If $r = 2\sin(\theta)$, then the area inside this curve is

$$\int_0^\pi \frac{1}{2}(2\sin(\theta))^2 d\theta = 2\int_0^\pi \sin^2(\theta) d\theta = \pi,$$

as expected, since r traces out a circle of radius 1. Notice if 2π was an upper limit of integration, we would get the wrong answer, since the map $r = 2\sin(\theta)$ if $\theta \in [0, 2\pi]$ traces out the circle twice.

Exercise 4.28. Find the following area of

(a) The region enclosed by the graph $r = 1 + \frac{1}{2}\cos(\theta)$.

(b) One petal of the rose $r = \cos(3\theta)$.

(c) The inner loop of the limaçon $r = 1 - 2\cos(\theta)$.

(d) The interior of the limaçon $r = 1 + b\sin(\theta)$ if $0 \leq b \leq 1$.

(e) The region outside of the cardioid $r = 1 + \cos(\theta)$ and inside the circle $r = 3\cos(\theta)$. [Strang, pg 359]

(f) The region between the graph of $r = e^{-\theta}$ if $\theta \geq 0$ and the origin. [Strang, pg 359]

4.3 Polar form of complex numbers

We consider complex numbers, $a + bi$, with $a, b \in \mathbb{R}$ and $i^2 = -1$. These correspond to ordered pairs of real numbers. $a+bi$ corresponds to (a, b). So, geometrically, the set of all complex numbers corresponds to the rectangular plane. In this context, it's called the complex plane and denoted by $\mathbb{C}$. Complex numbers were invented to solve cubic polynomial equations. And they have nice arithmetic properties.

In particular, we can add them.

$$(a + bi) + (c + di) = a + c + (b + d)i.$$

We can also multiply them.

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

Geometrically, the **norm** of a complex number is its distance from the origin. If $z = a + bi$, then the norm of z is $|z| = \sqrt{a^2 + b^2}$. Notice, if $z \neq 0 + 0i$, then $|z| > 0$.

Recall, in polar coordinates, $(x, y) = (r\cos(\theta), r\sin(\theta))$. So, we should be able to write any complex number z in polar coordinates as

$$z = r\cos(\theta) + ir\sin(\theta)$$

with $r = |z|$ and $\theta = \arg(z)$, which is the angle between z and the polar axis. We introduce the **complex exponential**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for every $\theta \in \mathbb{R}$. Then we can write any z in **polar form** as

$$z = e^{\ln(r)+i\theta}$$

with $r = |z|$ and $\theta = \arg(z)$. Writing θ in the exponent works because of the sum rules for cos and sin. Namely, if $|z_i| = r_i$ and $\arg(z_i) = \theta_i$ for $i = 1, 2$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$

$$
\begin{aligned}
&+(\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2))i) \\
&= r_1 r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)) \\
&= r_1 r_2 e^{i(\theta_1+\theta_2)}.
\end{aligned}
$$

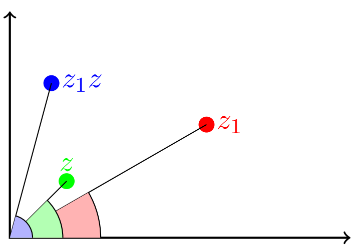
This is a nice way to describe complex multiplication. If $z_1 = r_1 e^{i\theta_1}$, then we can define a function lm_{z_1}, left multiplcation by z_1, as a function from $\mathbb{C}$ to $\mathbb{C}$, for which

$$\mathrm{lm}_{z_1}(z) = z_1 z$$

for every $z \in \mathbb{C}$. If $z = re^{i\theta}$, then

$$\mathrm{lm}_{z_1}(z) = r_1 r e^{i(\theta_1+\theta)}.$$

This means multiplying z on the left by z_1 has the result of scaling z by $|z_1|$ and rotating z by $\arg(z_1)$.



Exercise 4.29.

In the image, $z = (0.5, 0.5)$ and $z_1 = (\sqrt{3}, 1)$. Find the polar form of $z_1 z$.

We can also divide complex numbers easily. Recall the complex conjugate of a complex number $a+bi$ is $a-bi$, the reflection across the x-axis. If $z = a+bi$, we denote the complex conjugate of z by $\overline{z} = a-bi$. We have

$$z \cdot \frac{\overline{z}}{|z|^2} = 1.$$

If $z = re^{i\theta}$, then

$$\overline{z} = re^{-i\theta}.$$

Hence,

$$z \cdot \frac{\overline{z}}{|z|^2} = re^{i\theta} \cdot \frac{re^{-i\theta}}{r^2} = e^{i(\theta-\theta)} = e^0 = 1.$$

That is, the multiplicative inverse of z is $\dfrac{\overline{z}}{|z|^2}$. Hence,

$$\frac{z_1}{z_2} = z_1 \cdot \frac{\overline{z_2}}{|z_2|^2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)}.$$

We have

Theorem 4.30. (De Moivre's formula) If n is a positive integer, then

$$(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta)+i\sin(n\theta)$$

for all real θ.

Proof. We will prove this by induction. The base case, $n = 1$, is trivial. Inductive Step: If

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta),$$

then

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^{n+1} &= (\cos(\theta) + i\sin(\theta))^n(\cos(\theta) + i\sin(\theta)) \\ &= (\cos(n\theta) + i\sin(n\theta))(\cos(\theta) + i\sin(\theta)) \\ &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) + i(\cos(\theta)\sin(n\theta) + \sin(\theta)\cos(n\theta)) \\ &= \cos((n+1)\theta) + i\sin((n+1)\theta).\end{aligned}$$

By induction, we've proved the result. □

This allows us to exponentiate complex numbers with ease. If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$, which is also consistent with arithmetic of exponents: $(ab)^s = a^s b^s$ and $(a^s)^t = a^{st}$.

Example 4.31. For example, suppose we wish to find

$$(-1 - i)^5.$$

Then we write $-1-i$ in polar form using $r = \sqrt{2}$ and $\theta = \arctan 2(-1, -1) = \arctan(1) - \pi = -\frac{3\pi}{4}$. Then

$$z = \sqrt{2}e^{-i\frac{3\pi}{4}},$$

so that

$$z^5 = 2^{\frac{5}{2}}e^{-i\frac{15\pi}{4}} = \sqrt{32}e^{i\frac{\pi}{4}-4\pi} = \sqrt{32}e^{i\frac{\pi}{4}} = 4 + 4i.$$

Exercise 4.32. Write the following complex numbers in polar form:

1. $(1 + 2i)^2$
2. $(2 + i)^3$
3. $(3 - i)^4$.

We can also take nth roots of complex numbers.

Theorem 4.33. If n is a positive integer, $r > 0$, and $z^n = re^{i\theta}$, then

$$z = r^{\frac{1}{n}}e^{i(\frac{\theta}{n}+\frac{2\pi k}{n})}$$

for some integer $k \in \{0, 1, 2, \ldots, n-1\}$.

Proof. If $z^n = re^{i\theta}$, and $z = se^{it}$ for some real $s > 0$ and t, then

$$s^n e^{int} = re^{i\theta}.$$

This implies

$$\frac{s^n}{r} = e^{i(\theta - nt)}.$$

Since r, s, θ, t are all real numbers, $s^n/r > 0$, and $e^{i(\theta-nt)} = \cos(\theta - nt) + i\sin(\theta - nt)$, it follows $\theta - nt = 2\pi l$ for some integer l, because the imaginary part must vanish and $\cos(\theta - nt) > 0$. We have $l = mn - k$ for some integer m and unique $0 \leq k \leq n-1$. This is the division algorithm. Then

$$t = \frac{\theta}{n} + \frac{2\pi k}{n} - 2\pi m.$$

Also since $\theta - nt = 2\pi l$, it follows $s^n = r$. That is,

$$s = r^{\frac{1}{n}}.$$

Finally,

$$z = se^{it} = r^{\frac{1}{n}} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n} - 2\pi m)} = r^{\frac{1}{n}} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})}.$$

□

Example 4.34. Let's find the **cube roots of unity**. That is, let's find the complex solutions of

$$z^3 = 1.$$

We write

$$1 = e^{i0}.$$

Thus, if $z^3 = 1$, then either

$$z = 1,$$

$$z = e^{i\frac{2\pi}{3}},$$

or

$$z = e^{i\frac{4\pi}{3}}.$$

In general, the solutions of $z^n = 1$ are called the **nth roots of unity**. Notice there are n of them, consistent with the Fundamental Theorem of Algebra.

Exercise 4.35.

(a) Find the fourth roots of unity.

(b) Find the third roots of -2.

(c) Find the square roots of $1 + i$.

A Mathematical Induction

Definition A.1. The **Principle** of **Mathematical Induction** states, for any subset S of $\mathbb{N}$, if $0 \in S$ and, for all $n \in \mathbb{N}$, $n \in S$ implies $n+1 \in S$, then $S = \mathbb{N}$. Formally, for any subset S of $\mathbb{N}$, we assume the following statement

$$(0 \in S \wedge (\forall n \in \mathbb{N})\,(n \in S \Rightarrow n+1 \in S)) \Rightarrow S = \mathbb{N}$$

Definition A.2. The **Principle** of **Strong Mathematical Induction** states, for any subset S of $\mathbb{N}$, if, for all $n \in \mathbb{N}$, $\{0, \ldots, n\} \subseteq S$ implies $n+1 \in S$, then $S = \mathbb{N}$. Formally, for any subset S of $\mathbb{N}$, we assume the following statement

$$(\forall n \in \mathbb{N})\,(\{0, \ldots, n\} \subseteq S \Rightarrow n+1 \in S) \Rightarrow S = \mathbb{N}$$

Theorem A.3. The Principle of Mathematical Induction and the Principle of Strong Mathematical Induction are equivalent.

Proof. Suppose the Principle of Mathematical Induction is true. Suppose, for all $n \in \mathbb{N}$, $\{1, \dots, n\} \subseteq S$ implies $n+1 \in S$. Then $0 \in S$ and, for all $n \in \mathbb{N}$, $n \in S$ implies $n+1 \in S$. Hence, by the Principle of Mathematical Induction $S = \mathbb{N}$.

Suppose the Principle of Strong Mathematical Induction is true. Suppose $1 \in S$ and, for all $n \in \mathbb{N}$, $n \in S$ implies $n+1 \in S$. Then for all $n \in \mathbb{N}$, $\{0, \dots, n\} \subseteq S$ implies $n+1 \in S$. Hence, by the Principle of Strong Mathematical Induction $S = \mathbb{N}$. □

Definition A.4. The **Well Ordering Principle** states, for any nonempty subset S of $\mathbb{N}$, there exists a minimum element of S. Formally, for any subset S of $\mathbb{N}$, we assume the following statement

$$S \neq \emptyset \Rightarrow (\exists m \in S)(\forall n \in S)(m \leq n)$$

Theorem A.5. The Principle of Mathematical Induction and the Well Ordering Principle are equivalent.

Proof. Suppose the Principle of Mathematical Induction is true. Then by the previous theorem, the Principle of Strong Mathematical Induction is true. Suppose S is a subset of $\mathbb{N}$. We wish to prove S is either empty or has a least element.

Consider

$$\mathbb{N} \setminus S = \{m \in \mathbb{N} \mid m \notin S\}.$$

$\mathbb{N} \setminus S$ is called the complement of S with respect to $\mathbb{N}$.

We will consider a few cases. The following statement is true.

$$\text{Either } 0 \in S \text{ or } 0 \in \mathbb{N} \setminus S.$$

If $0 \in S$, then S is nonempty and has a minimum element, namely, 0. Now we move on to the case $0 \in \mathbb{N} \setminus S$. This case has a few subcases. The following statement is true.

Either, for all $n \in \mathbb{N}, \{0, \dots, n\} \subseteq \mathbb{N} \setminus S$ implies $n+1 \in \mathbb{N} \setminus S$,
or there is some $n \in \mathbb{N}$ such that $\{0, \dots, n\} \subseteq \mathbb{N} \setminus S$ and $n+1 \notin \mathbb{N} \setminus S$.

If, for all $n \in \mathbb{N}, \{0, \dots, n\} \subseteq \mathbb{N} \setminus S$ implies $n+1 \in \mathbb{N} \setminus S$, then since $0 \in \mathbb{N} \setminus S$ by assumption, by the Principle of Strong Mathematical Induction, $\mathbb{N} \setminus S = \mathbb{N}$. Hence, $S = \mathbb{N} \setminus (\mathbb{N} \setminus S) = \mathbb{N} \setminus \mathbb{N} = \emptyset$. In other words, S is empty. Lastly, if there is some $n \in \mathbb{N}$ such that $\{0, \dots, n\} \subseteq \mathbb{N} \setminus S$ and $n+1 \notin \mathbb{N} \setminus S$, then $n+1 \in S$, and $n+1$ is the minimum element of S, because all other natural numbers less than $n+1$ are not in S. Hence, in any case, S is either empty or has a minimum element. We have proven the Well Ordering Principle from the Principle of Mathematical Induction.

Now suppose the Well Ordering Principle is true and consider a set S such that $0 \in S$, and $n \in S$ implies $n+1 \in S$ for all $n \in \mathbb{N}$ but contrarily to Induction, $S \neq \mathbb{N}$. Then the complement $\mathbb{N} \setminus S$ is nonempty and, hence, by the Well-Ordering Principle, has a minimum element, say $m \in \mathbb{N} \setminus S$. But then $m-1 \in S$ because $m-1 < m$. From our assumption on S, $m = m-1+1 \in S$. But m can't be both an element of S and an element of its complement. This is a contradiction. Our assumption that $S \neq \mathbb{N}$ must be false. Hence, $S = \mathbb{N}$ given $0 \in S$, and $n \in S$ implies $n+1 \in S$ for all $n \in \mathbb{N}$. We have proven the Principle of Mathematical Induction from the Well Ordering Principle.

□

Well Ordering is taken as fact without justification because it seems intuitive. Or at least it's a property we expect a set such as what we think of the natural numbers to have. With the assumption of Well Ordering, the previous theorem tells us that the assumption of Mathematical Induction is just as justified. We usually employ Induction as a proof technique for statements of the form: for every $n \in \mathbb{N}, \phi(n)$, if $\phi(n)$ is some formula with variable n. This is because proving the statement

$$\forall n \phi(n)$$

is equivalent to proving

$$\{n \in \mathbb{N} \mid \phi(n)\} = \mathbb{N}\,.$$

So, if we prove

$$\phi(0) \wedge (\forall n \in \mathbb{N})(\phi(n) \Rightarrow \phi(n+1)),$$

then by Mathematical Induction,

$$\forall n \phi(n)$$

Theorem A.6. For every n,

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$

Proof. We need to show this equation holds for 0 and holds for $n+1$ if it holds for n.

$$\sum_{k=0}^{0} k = 0$$

and

$$\frac{0 \cdot (0+1)}{2} = 0.$$

Hence, this equation holds for 0.

If

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2},$$

(if this equation holds for n) then

$$\sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+1+1)}{2}.$$

(this equation holds for $n+1$)

Hence, by induction, for every n,

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$

□

Definition A.7. For each natural number n and k, with $k \leq n$, the number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is called the **binomial coefficient**. Here and elsewhere, for any natural number n,

$$\begin{aligned} 0! &= 1 \\ n! &= n \cdot (n-1)!. \end{aligned}$$

$n!$ is the **factorial** of n. It's the successive product of n with all smaller natural numbers.

For example, the previous theorem claims

$$\sum_{k=0}^{n} k = \binom{n+1}{2}.$$

Exercise A.8. For any natural numbers n and k with $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{A.9}$$

This does not need induction. It can be shown directly.

Theorem A.10. The Binomial Theorem

For any natural number n, for any nonzero real numbers a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Proof. We prove this statement by induction. Fix arbitrary nonzero $a, b \in \mathbb{R}$. Base case: If $n = 0$, then

$$(a+b)^0 = 1.$$

And

$$\sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^k = \binom{0}{0} a^{0-0} b^0 = \frac{0!}{0!(0-0)!} = 1.$$

Inductive hypothesis: Suppose n is arbitrary other than

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Inductive step:

$$\begin{aligned} (a+b)^{n+1} = (a+b)(a+b)^n &= (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \end{aligned}$$

Now we perform a operation called **shifting the index**. It's like a change of variable. If $l = k+1$, then $k = l-1$, and if $k = 0$, then $l = 1$ and if $k = n$, then $l = n+1$. Then

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{l=1}^{n+1} \binom{n}{l-1} a^{n+1-l} b^{l}.$$

It doesn't matter what we call the index. We can replace l with k again to conclude

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^{k} + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\
&\qquad = \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^{k} \\
&\qquad = a^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) a^{n+1-k} b^{k} + b^{n+1}.
\end{aligned}$$

Now we can use (A.9) to conclude

$$\begin{aligned}
(a+b)^{n+1} &= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^{k} + b^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^{k}.
\end{aligned}$$

The last step holds in part because $\binom{n+1}{0} = 1 = \binom{n+1}{n+1}$.

Hence, we proved, for all $n \in \mathbb{N}$,

$$\text{if } (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}, \text{ then } (a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^{k}$$

This, together with

$$(a+b)^0 = \sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^{k}$$

and the Principle of Mathematical Induction, proves the statement for all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}.$$

□

B Cauchy Sequences and the Completeness of the Real Number System

Definition B.1. A sequence $\{a_n\}_{n=0}^{\infty}$ is **Cauchy** if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq N$.

Theorem B.2. If a sequence of real numbers converges, then it's Cauchy.

Proof. Suppose $\lim_{n\to\infty} a_n = a$. Then, for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ for all $n \geq N$. Then if $m, n \geq N$, then

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon\,.$$

Hence, $\{a_n\}_{n=0}^{\infty}$ is Cauchy. □

Definition B.3. (Completeness of the reals) If a sequence of real numbers is Cauchy, then it converges to a real number.

Just like the Well-Ordering Principle of the Natural Numbers, the previous statement for any sequence of real numbers is one we expect the Real Number System to have. When constructing the reals this definition becomes a theorem. But it's enough for our purposes to take the so-called synthetic approach.

Definition B.4.

(a) An **upper bound** of a subset S is a number $b \in \mathbb{R}$ such that $s \leq b$ for all $s \in S$.

(b) A **least upper bound** or **supremum** of a subset S is a number $r \in \mathbb{R}$ such that r is an upper bound of S and, for any upper bound b of S, $r \leq b$. We denote the supremum of a subset S by $\sup S$.

(c) An **lower bound** of a subset S is a number $b \in \mathbb{R}$ such that $b \leq s$ for all $s \in S$.

(d) A **greatest lower bound** or **infimum** of a subset S is a number $r \in \mathbb{R}$ such that r is an lower bound of S and, for any lower bound b of S, $b \leq r$. We denote the infimum of a subset S by $\inf S$.

(e) We extend the definitions of sup and inf to include unbounded S: $\sup S = \infty$ if S has no upper bound and $\inf S = -\infty$ if S has no lower bound. And we define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$, with $\emptyset$ is the set with no members.

Exercise B.5.

(a) There is at most one upper bound for every subset of the real numbers.

(b) x is the least upper bound of a subset S if and only if x is an upper bound of S and for all $\varepsilon > 0$, there exists some s in S for which $x - \varepsilon \leq s$. ε is the Greek letter epsilon.

(c) Describe the square root of 2 as the least upper bound of a subset of rational numbers.

Theorem B.6. (Characterization of completeness) Every nonempty subset of $\mathbb{R}$ with an upper bound has a real least upper bound.

C The Riemann Integral

Definition C.1. We suppose f is a bounded real-valued function defined on the closed and bounded interval $[a, b]$.

(a) A **partition** of $[a, b]$ is a finite set of real numbers $x_0, \ldots, x_n$ such that $a = x_0 < \cdots < x_n = b$.

(b) The **lower Darboux sum** of f with respect to P is

$$L(f, P) := \sum_{i=1}^{n} \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}(x_i - x_{i-1}).$$

(c) The **upper Darboux sum** of f with respect to P is

$$U(f, P) := \sum_{i=1}^{n} \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}(x_i - x_{i-1}).$$

(d) The **lower Riemann integral** of f on $[a, b]$ is

$$L(f, [a, b]) = \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$

(e) The **upper Riemann integral** of f on $[a, b]$ is

$$U(f, [a, b]) = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

(f) If $L(f, [a, b]) = U(f, [a, b])$, we say f is **Riemann integrable** and denote this common value by $\int_a^b f$ or $\int_a^b f(x)dx$.

Theorem C.2. If $a, b \in \mathbb{R}$, f, g are Riemann integrable, then

(a) $\int (af + bg) = a \int f + b \int g$

(b) $|\int f| \leq \int |f|$.