College Algebra Notes

Incomplete and full of errors.
Subject to change.

Contents

1 Some Arithmetic and Geometry of Real Numbers

Real numbers are those numbers which we can add, multiply and order them in such a way that the collection of them fill up a line.

When two real numbers a and b are **equal**, we write $a = b$. Somewhat circularly, we say two real numbers are equal exactly when their difference is zero.

Definition 1.1. The $=$ symbol has the following properties. Suppose a, b, c are real numbers.

(a) If $a = b$, then $b = a$. (symmetry)

(b) $a = a$. (reflexivity)

(c) If $a = b$ and $b = c$, then $a = c$. (transitivity)

1.1 arithmetic and geometry

Definition 1.2. There is an operation on the real numbers, called **addition**, denoted $+$, whose input is two real numbers, and whose output is one real number. For any real numbers a and b, we write $a + b$ to denote the output (the **sum**) of a and b under $+$.

Addition is almost determined by the following properties.

For any three real numbers a, b, c,

$$a + (b + c) = (a + b) + c \qquad \textit{associativity}$$
$$a + b = b + a \qquad \textit{commutativity}$$

Associativity of addition means we can write expressions such as $a + b + c$ without ambiguity. Moreover, there exists a real number, denoted by 0, for which, for any real number a,

$$a + 0 = a \qquad 0 \textit{ is the additive identity}$$

And for any real number a, there is exactly one number, denoted by $-a$, for which

$$a + (-a) = 0 \qquad \textit{existence of additive inverses}$$

Notice commutativity implies also $0 + a = a$ and $-a + a = 0$ for every real number a.

Finally,

$$\textit{if } a = b \textit{ then } a + c = b + c \qquad \textit{addition preserves equality}$$

And commutativity also implies if $a = b$, then $c + a = c + b$.

We define **subtraction** in terms of addition as follows. For any two real numbers a and b,

$$a - b := a + (-b).$$

The := means we are introducing new notation for convenience on the side of :. ■

Exercise 1.3. Explain why, if a and b are real numbers, then

$$-(a - b) = b - a.$$

Definition 1.4. There is an operation on the real numbers, called **multiplcation**, denoted $\cdot$, whose input is two real numbers, and whose output is one real number. For any real numbers a and b, we write $a \cdot b$ to denote the output (the **product**) of a and b under $\cdot$.

Multiplication is almost determined by the following properties.

For any three real numbers a, b, c,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \textit{associativity}$$

$$a \cdot b = b \cdot a \qquad \textit{commutativity}$$

Associativity of multiplication means we can write expressions such as $a \cdot b \cdot c$ without ambiguity. Moreover, there exists a real number, denoted by 1, for which

$$1 \neq 0 \qquad 1 \textit{ is not equal to } 0$$

and for any real number a,

$$1 \cdot a = a \qquad 1 \textit{ is the multiplicative identity}$$

And for any real number a which isn't zero, there is exactly one number, denoted by a^{-1}, for which

$$a^{-1} \cdot a = 1 \qquad \textit{existence of multiplicative inverses}$$

Notice commutativity implies also $a \cdot 1 = a$ and $a \cdot a^{-1} = 1$ for every real number a.

Finally,

$$\textit{if } b = c \textit{ then } a \cdot b = a \cdot c \qquad \textit{multiplication preserves equality}$$

And commutativity also implies if $b = c$, then $b \cdot a = c \cdot a$.

Sometimes we denote $a \cdot b$ just by ab, $a(b)$ or $(a)b$ for any real numbers a and b if the context is clear.

We define **division** in terms of multiplication as follows. For any two real numbers, a and b, if b is not zero,

$$\frac{a}{b} := a \cdot b^{-1}.$$

■

Lemma 1.5.

(a) If a and b are any real numbers, then $b + (-b) + a = a$ and $a + b + (-b) = a$.

(b) If a and b are real numbers and $a \neq 0$, then $aa^{-1}b = b$ and $baa^{-1} = b$.

Proof. (a) Suppose a and b are real numbers. Then we know

$$b + (-b) = 0$$

since $-b$ is the additive inverse of b. Hence,

$$b + (-b) + a = 0 + a$$

since addition preserves equality. Also,

$$0 + a = a$$

since 0 is the additive identity. Hence,

$$b + (-b) + a = a$$

by transitivity of equality. Then since

$$a + b + (-b) = b + (-b) + a$$

by commutativity of addition,

$$a + b + (-b) = a$$

follows from transitivity of equality.

(b) Suppose a and b are real numbers and $a \neq 0$. Then we know

$$aa^{-1} = 1$$

since a^{-1} is the multiplicative inverse of a. Then

$$baa^{-1} = b \cdot 1$$

since multiplication preserves equality. Then since

$$aa^{-1}b = baa^{-1}$$

by commutativity of multiplication, it follows

$$aa^{-1}b = b \cdot 1$$

by transitivity of equality. Now,

$$b \cdot 1 = b$$

since 1 is the multiplicative identity, it follows

$$aa^{-1}b = b$$

by transitivity of equality. Now

$$baa^{-1} = aa^{-1}b$$

by commutativity of multiplication. Hence,

$$baa^{-1} = b$$

by transitivity of equality.

□

Theorem 1.6. Properties of Algebra
For any three real numbers a, b, c,

(a) If $a + c = b + c$, then $a = b$.

(b) If a is not zero and $ab = ac$, then $b = c$.

Proof. Let a, b, c be arbitrary.

(a) Suppose

$$a + c = b + c.$$

Then

$$a + c + (-c) = b + c + (-c) \tag{1.7}$$

since addition preserves equality. Now, by Lemma 1.5, it follows

$$a + c + (-c) = a.$$

Then

$$a = a + c + (-c) \tag{1.8}$$

by symmetry of equality. Hence, by equations 1.7 and 1.8, it follows

$$a = b + c + (-c)$$

by transitivity of equality. We also know

$$b + c + (-c) = b$$

by Lemma 1.5. Hence,

$$a = b$$

by transitivity of equality.

(b) This is proved similarly as in (a) only using a^{-1} and 1 instead of $-c$ and 0, keeping in mind that $a \neq 0$.

□

Exercise 1.9.

(a) Explain why $-0 = 0$.

(b) Explain why $1^{-1} = 1$.

(c) Explain why $(a^{-1})^{-1} = a$ for all nonzero real numbers a.

(d) Explain why $(ab)^{-1} = b^{-1}a^{-1}$ for all nonzero real numbers a and b.

(e) Explain why, for any two real numbers a and b, if $a + b = a$, then $b = 0$.

(f) Explain why, for any two real numbers a and b, if $a \cdot b = b$, then $a = 1$.

(g) Explain why the following statement is either true or false. If a is any real number, then $\frac{1}{\frac{1}{a}} = a$.

Definition 1.10. Addition and multiplication are completely determined by the above definitions and the following property which ties them together.

For any three real numbers a, b, c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \qquad \textit{distributivity}$$

Commutativity of multiplication also implies

$$(b + c) \cdot a = ab + ac$$

for any three real numbers a, b, c.

In this last equation and elsewhere, we follow the conventional order of operations. The operation inside the innermost parenthesis is performed first, then multiplication, then addition. Thus, when computing $a \cdot (b + c)$, we first compute $b + c$, then multiply a with that result. Distribution says this number is equal to first finding ab and ac separately, then adding the results.

■

Exercise 1.11.

(a) Prove if a and b are real numbers, that

$$(-a)b = -(ab)$$

and

$$(-a)(-b) = ab.$$

(b) Prove if a is a real number, then $(-1)a = -a$.

(c) Prove $-(-a) = a$ for all real numbers a.

(d) Explain why $-(a^{-1}) = (-a)^{-1}$ for all nonzero real numbers a.

(e) Prove if a is a real number, then $a \cdot 0 = 0$.

(f) Explain why 0 does not have a multiplicative inverse.

(g) Explain, using an example, why it is not always true that if $ab = ac$ for some real numbers a, b, c, then $b = c$ necessarily.

Definition 1.12. We now define what it means for a number to be **positive**. The collection of positive numbers is determined by the following three properties.

For any number a, either

$$\left.\begin{array}{r} a \textit{ is positive} \\ a = 0 \\ -a \textit{ is positive} \end{array}\right\} \qquad \textit{trichotomy}$$

For any two numbers a and b,

if a and b are positive, then $a + b$ is positive — *closure under addition*

For any two numbers a and b,

if a and b are positive, then ab is positive — *closure under multiplication*

We introduce the relation $<$ and say, for any two numbers a, b,

$$a < b$$

(read a is less than b) if $b - a$ is positive. Another way of saying a is positive is to write in symbols $0 < a$. Then, $a < b$ if and only if $0 < b - a$. Since $-(b - a) = a - b$ by Exercise 1.3, if $-(b - a)$ is positive, we sometimes write $b - a < 0$ instead of $b < a$. If a number is neither positive nor zero, we call it **negative**. If a number is either positive or zero, we call it **nonnegative**.

If a is nonnegative, we write $0 \le a$ to mean either $0 < a$ or $a = 0$. We also say $a \le b$ to mean $0 \le b - a$ and say a is less than or equal to b. We also write $b > a$ to mean $a < b$ and say b is greater than a. And write $b \ge a$ to mean $a \le b$ and say b is greater than or equal to a.

$<$ is called an **order relation**.

■

Theorem 1.13. For any three real numbers a, b, c,

(a) (Trichotomy of order) Exactly one of the following holds. Either

  (i) $a < b$

  (ii) $a = b$ or

  (iii) $b < a$.

(b) (Transitivity of order) If $a < b$ and $b < c$, then $a < c$. $a < b$ and $b < c$ is usually written concisely as $a < b < c$.

(c) (Addition preserves order) If $a < b$, then $a + c < b + c$.

(d) (multiplication by a positive number preserves order) If $0 < a$ and $b < c$, then $ab < ac$.

Proof. (a) If is it not the case that $a < b$, then it is not the case that $b - a$ is positive. Hence, either $b - a = 0$ or $0 < -(b - a)$. Hence, if $b - a = 0$, then $0 = b - a$ by symmetry of equality. Then $0 + a = b - a + a$ since addition preserves equality. Then $0 + a = a$ since 0 is the additive identity. Hence,

$$a = 0 + a$$

by symmetry of equality. Hence,

$$a = b - a + a$$

by transitivity of equality. Since $b - a + a = b$ by Lemma 1.5, it follows

$$a = b$$

by transitivity of equality. If $b - a \neq 0$, then $0 < -(b - a)$. Since $-(b - a) = a - b$ by Exercise 1.3, it follows $0 < a - b$. That is, by definition, $b < a$. In conclusion, exactly one of the following holds. Either $a < b$, $a = b$ or $b < a$.

(b) If $a < b$ and $b < c$, then $0 < b - a$ and $0 < c - b$. Hence, since positivity is closed under addition, $0 < b - a + c - b$. Since $b - b + a - c = b - a + c - b$ by commutativity of addition and $a - c = b - b + a - c$ by Lemma 1.5 and symmetry of equality, it follows $a - c = b - a + c - b$ by transitivity of equality. That is, since $b - a + c - b = a - c$ and $0 < b - a + c - b$, it follows $0 < a - c$. That is, $a < c$. In conclusion, we've shown if $a < b < c$, then $a < c$.

(c) If $a < b$, then $0 < b - a$. Since $b - a = b - a + c - c$ by Lemma 1.5 and $b - a + c - c = b + c - a - c$ by commutativity of addition, it follows $b - a = b + c - a - c$ by transitivity of equality. Further, $-a = -1 \cdot a$ and $-c = -1 \cdot c$ by Exercise 1.11 (b). Hence,

$$-a - c = -1 \cdot a - c$$

since addition preserves equality. And

$$-1 \cdot a - c = -1 \cdot a + -1 \cdot c$$

since addition preserves equality. Hence,

$$-a - c = -1 \cdot a + -1 \cdot c$$

by transitivity of equality. Then

$$b + c - a - c = b + c - 1 \cdot a + -1 \cdot c$$

since addition preserves equality. Now,

$$-1 \cdot a + -1 \cdot c = -1(a + c)$$

by distributivity. Hence,

$$b + c - 1 \cdot a + -1 \cdot c = b + c - 1(a + c)$$

since addition preserves equality. And

$$-1(a+c) = -(a+c)$$

by Exercise 1.11 (b). Hence,

$$b+c-1(a+c) = b+c-(a+c)$$

since addition preserves equality Finally, since

$$b-a = b+c-a-c,$$

it follows by repeated use of transitivity of equality, that

$$b-a = b+c-(a+c).$$

Hence, since $0 < b-a$, it follows $0 < b+c-(a+c)$. That is, by definition, $a+c < b+c$.

(d) If $0 < a$ and $b < c$, then $0 < c-b$, and since positivity is closed under multiplication, it follows $0 < a(c-b)$. Now, $a(c-b) = ac - ab$ by distribution. Hence,

$$0 < ac - ab.$$

That is,

$$ab < ac$$

by definition.

□

Exercise 1.14. (a) Show, if $a < 0$ and $b < 0$, then $a+b < 0$.

(b) Show, if $a < 0$ and $b < 0$, then $0 < ab$.

(c) Also, if $0 < a$ and $b < 0$, then $ab < 0$.

(d) Explain why if $ab = 0$, then either $a = 0$ or $b = 0$. This is called the Zero Product Property.

(e) Show $0 < 1$.

(f) Show $0 < 1+1$.

(g) If a is a real number and $a > 1$, then $0 < a^{-1} < 1$.

(h) If $a \leq b$ and $b \leq a$, then $a = b$.

(i) If $a < 0$ and $b < c$, then $ac < ab$.

Definition 1.15. The **absolute value** of a real number a is defined to be a if $a \geq 0$ and $-a$ if $a < 0$. The absolute value of a is denoted by $|a|$. Notice $0 \leq |a|$ for all real numbers a and $|a| = 0$ only when $a = 0$.

■

The absolute value, together with the following theorem and formally the order relation itself, provide the geometric backdrop of the real number system.

Theorem 1.16. The triangle inequality
If a and b are real numbers, then $|a+b| \leq |a|+|b|$.

Proof. Let a and b be real numbers. Remember, by trichotomy of positivity, we only have a few cases to check.

1. If $a \geq 0$ and $b \geq 0$, then $|a| = a$ and $|b| = b$. Also, if $a \geq 0$ and $b \geq 0$, then $a+b \geq 0$. So, $|a+b| = a+b$. Hence,
$$|a+b| = a+b = |a|+|b|.$$
In particular, in this case,
$$|a+b| \leq |a|+|b|.$$

2. If $a < 0$ and $b < 0$, then $|a| = -a$, $|b| = -b$. Also, if $a < 0$ and $b < 0$, then $a+b < 0$ by Exercise 1.14. So, $|a+b| = -(a+b)$. Hence,
$$|a+b| = -(a+b) = -a+(-b) = |a|+|b|.$$

3. If $a < 0$, $b \geq 0$ and $a+b < 0$, then $|a| = -a$, $|b| = b$ and $|a+b| = -(a+b)$. By the above exercise, $-b \leq |b|$. Hence,
$$|a+b| = -(a+b) = -a-b = |a|+(-b) \leq |a|+|b|.$$

4. If $a < 0$, $b \geq 0$ and $a+b \geq 0$, then $|a| = -a$, $|b| = b$ and $|a+b| = a+b$. By the above exercise, $-a \leq |a|$. Hence,
$$|a+b| = a+b = a+|b| \leq |a|+|b|.$$

5. The two other cases when $a \geq 0$ and $b < 0$ are identical to the last two items. Simply switch the roles of a and b.

□

Exercise 1.17.

(a) Explain with cases why, if a is a real number, then $a \leq |a|$.

(b) Explain, if, for any real number a, $\operatorname{sgn}(a) = 1$ if $a > 0$, $\operatorname{sgn}(a) = -1$ if $a < 0$ and $\operatorname{sgn}(a) = 0$ if $a = 0$, then $a = \operatorname{sgn}(a)|a|$ and $|a| = \operatorname{sgn}(a)a$.

(c) Explain, if a and b are real numbers, then $|ab| = |a||b|$.

(d) Explain, if a and b are real numbers, then $||a|-|b|| \leq |a-b|$.

(e) Explain with cases why, if a and b are real numbers, $b \geq 0$, and $-b \leq a \leq b$, then $|a| \leq b$. And if $|a| \leq b$, then $-b \leq a \leq b$.

We close with an application of the many facts we have gathered.

Theorem 1.18. For any two numbers a and b,

$$a - b = b - a$$

if and only if

$$a = b.$$

Proof. When we say P if and only if Q for two propositions P and Q, we mean the following. If P, then Q and if Q then P. Suppose a and b are any two real numbers. If

$$a - b = b - a,$$

then

$$a - b + b = b - a + b \tag{1.19}$$

since addition preserves equality. We have

$$a - b + b = a$$

by Lemma 1.5. Then

$$a = a - b + b \tag{1.20}$$

by symmetry of equality. Hence, by Equations 1.19 and 1.20, it follows

$$a = b - a + b$$

by transitivity of equality. Now

$$b - a + b = b + b - a$$

by commutativity of addition. Hence,

$$a = b + b - a$$

by transitivity of equality. Hence,

$$a + a = b + b - a + a \tag{1.21}$$

since addition preserves equality. Again by Lemma 1.5, it follows

$$b + b - a + a = b + b + 0.$$

And

$$b + b + 0 = b + b \tag{1.22}$$

since 0 is the additive identity. Hence,

$$b + b - a + a = b + b \tag{1.23}$$

by transitivity of equality. Hence, by Equations 1.21 and 1.23, it follows

$$a + a = b + b \tag{1.24}$$

by transitivity of equality. Now we know

$$1 \cdot a = a \tag{1.25}$$

since 1 is the multiplicative identity. Hence,

$$1 \cdot a + 1 \cdot a = a + 1 \cdot a$$

since addition preserves equality. And again by Equation 1.25

$$1 \cdot a + a = a + a$$

since addition preserves equality. Then

$$a + 1 \cdot a = 1 \cdot a + a$$

by commutativity of addition. Then

$$a + 1 \cdot a = a + a$$

by transitivity of equality. Then since

$$1 \cdot a + 1 \cdot a = a + 1 \cdot a,$$

it follows

$$a + 1 \cdot a = 1 \cdot a + 1 \cdot a$$

by symmetry of equality. Hence,

$$1 \cdot a + 1 \cdot a = a + a$$

by transitivity of equality. We know

$$1 \cdot a + 1 \cdot a = (1 + 1) \cdot a$$

by distributivity. Hence,

$$(1 + 1) \cdot a = 1 \cdot a + 1 \cdot a$$

by symmetry of equality. It follows

$$(1 + 1) \cdot a = a + a$$

by transitivity of equality. Hence, by this equation and 1.24, it follows

$$(1 + 1) \cdot a = b + b$$

by transitivity of equality. Similarly, we may show

$$b + b = (1 + 1) \cdot b.$$

Hence,

$$(1 + 1) \cdot a = (1 + 1) \cdot b$$

by transitivity of equality. Since $1 + 1 \neq 0$ by Exercise 1.14 and trichotomy of positivity, it follows

$$(1 + 1)^{-1}(1 + 1)a = a$$

by Lemma 1.5. And

$$a = (1 + 1)^{-1}(1 + 1)a$$

by symmetry of equality. Hence, since $(1+1) \cdot a = (1+1) \cdot b$ implies

$$(1+1)^{-1}(1+1)a = (1+1)^{-1}(1+1) \cdot b$$

since multiplication preserves equality, it follows

$$a = (1+1)^{-1}(1+1) \cdot b$$

by transitivity of equality. Finally, since

$$(1+1)^{-1}(1+1) \cdot b = b$$

by Lemma 1.5, it follows

$$a = b$$

by transitivity of equality. In conclusion, if $a - b = b - a$, then $a = b$.

We now prove the converse. That is, if

$$a = b,$$

then

$$a - b = b - a.$$

Suppose

$$a = b.$$

Then

$$a - b = b - b.$$

Hence,

$$a - b = 0.$$

Also, if

$$a = b,$$

then

$$a - a = b - a.$$

Hence,

$$0 = b - a.$$

Hence, since $a - b = 0$ and $0 = b - a$, it follows

$$a - b = b - a.$$

$\square$

Notice we didn't mention the properties used to prove the converse. We just used them. We will do so from here on out and usually combine multiple properties in one step.

For example, to prove if $a - b = b - a$, then $a = b$, we may write if

$$a - b = b - a,$$

then

$$a = b + b - a.$$

Hence,

$$a + a = b + b.$$

Hence,

$$(1 + 1)a = (1 + 1)b.$$

Hence,

$$a = b.$$

Notice we condensed over thirty equations of proof into five. In these past few lemmas and theorems, we've been demonstrating the syntactic requirements of seemingly simple proofs. Indeed, the difference in explanations makes us wonder what we mean by the word proof. Maybe a proof is an explanation which can, if needed, be expanded to check the syntax of the string of statements and verified by computer. In practice, a proof of a claim should convince both the reader and the writer of the truthiness of that claim with little extra work.

1.2 The real numbers

Definition 1.26. If a and b are real numbers and $a < b$,

(a) the bounded open interval (a, b), is the set of all real numbers x for which $a < x < b$.

(b) the bounded closed interval $[a, b]$, is the set of all real numbers x for which $a \leq x \leq b$.

(c) the bounded half-open interval $(a, b]$, is the set of all real numbers x for which $a < x \leq b$.

(d) the bounded half-closed interval $[a, b)$, is the set of all real numbers x for which $a \leq x < b$.

The numbers a, b are called the **endpoints** of the intervals above. Notice an open interval doesn't contain its endpoints, a closed interval does, a half-open contains only its right endpoint, and a half-closed contains only its left.

If a is a real number,

(a) the unbounded open interval $(-\infty, a)$, is the set of all real numbers x for which $x < a$.

(b) the unbounded closed interval $(-\infty, a]$, is the set of all real numbers x for which $x \leq a$.

(c) the unbounded open interval (a, ∞), is the set of all real numbers x for which $a < x$.

(d) the unbounded closed interval $[a, \infty)$, is the set of all real numbers x for which $a \leq x$.

■

Theorem 1.27. The existence and uniqueness of positive nth roots

If n is a natural number and x is a positive real number, there exists a unique positive number y for which $y^n = x$. We denote such a y by $x^{\frac{1}{n}}$ and call this number the nth root of x.

For a proof, refer to Theorem B.16.

Definition 1.28.

(a) If $x > 0$, sometimes we denote $x^{\frac{1}{2}}$ by $\sqrt{x}$, called the **square root of** x.

(b) If n is a natural number, we define $0^{\frac{1}{n}} = 0$.

■

Definition 1.29.

(a) If m is a negative integer and x is any nonzero real number, then we define $x^m = (x^{-m})^{-1}$.

(b) If n is a negative integer and x is any positive real number, then we define $x^{\frac{1}{n}} = (x^{-1})^{-\frac{1}{n}}$.

(c) If m is an integer, n is a nonzero integer, and x is a positive real number, then we define $x^{\frac{m}{n}}$ by $(x^m)^{\frac{1}{n}}$.

■

Definition 1.30. Finally, if a is any real number, b is a positive real number and $b \neq 1$, there is a positive number, called **the** a**th power of** b and denoted by b^a. We say b^a is a **power** of b, a is the **exponent** of b^a, and b is the **base** of b^a.

The existence of such positive numbers can be justified by Definition B.21 and their properties are listed as Exercises in B.22.

Below is a drawing of the real numbers with some of its elements in relative position according to the ordering $<$.

2 Relations and Functions

2.1 Sets, ordered pairs, subsets, rectangular products, and relations

We assume the set of real numbers $\mathbb{R}$, as a collection of objects, exists with the properties listed in the previous section.

Definition 2.1. If n is a natural number and $a_1, \ldots, a_n$ are either real numbers or sets, we form their **set** by writing $\{a_1, \ldots, a_n\}$. For every $i = 1, \ldots, n$ a_i is called an **element** or **member** of the set $\{a_1, \ldots, a_n\}$. and write $a_i \in \{a_1, \ldots, a_n\}$. That is, the symbol $\in$ denotes membership. Two sets are **equal** if they contain the same elements. The order does not matter. If two sets S_1 and S_2 are equal, we sometimes write $S_1 = S_2$. If x and y are real numbers or sets, we define the **ordered pair** (x, y) to be the set $\{\{x\}, \{x, y\}\}$. If (x, y) is an ordered pair, x is called the **first coordinate** or **horizontal coordinate** and y the **second coordinate** or **vertical coordinate**.

As a note, the notation for the open interval (a, b) is unfortunately similar to the notation for the ordered pair (a, b). It will always be clear from context what we mean. ■

Definition 2.2. The **empty set** is the unique set which contains no elements, denoted by $\emptyset$. T is a **subset** of a set S if T is a set and, whenever t is an element of T, t is also an element of S.

If S is a set and $P(x)$ is a statement for any $x \in S$, we form the following subset of S in **set-builder notation**.

$$\{x \in S \mid P(x)\}.$$

$\{x \in S \mid P(x)\}$ is read as the set of all elements x in S such that $P(x)$ is true.

■

Exercise 2.3.

(a) Explain $\emptyset$ is a subset of any set.

(b) Explain, if S is a subset of T and T is a subset of S, then $S = T$.

(c) Show, if x, y, w, z are real numbers or sets, that $(x, y) = (w, z)$ as ordered pairs if and only if $x = w$ and $y = z$.

(d) Explain, if a, b are real numbers, then $(a, b) = (b, a)$ if and only if $a = b$.

Definition 2.4. If A and B are sets the **rectangular product** of A with B, denoted by $A \times B$, is the set of all ordered pairs (a, b) with a an element of A and b an element of B. A **relation** R between A and B is a subset of $A \times B$.

■

Exercise 2.5.

(a) Given any set A, find $\emptyset \times A$.

(b) Give an example to show the rectangular product between sets is not **associative**. That is, find three sets A, B, C, such that $(A \times B) \times C$ is not equal to $A \times (B \times C)$.

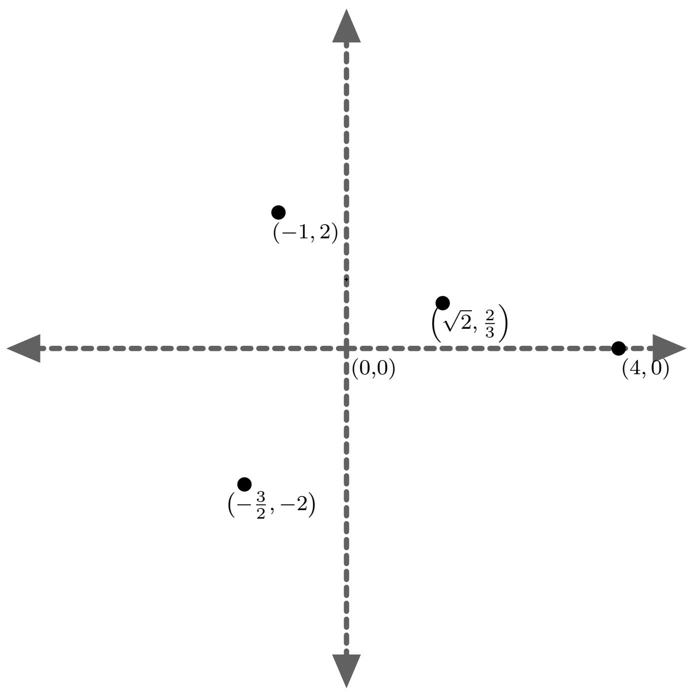
(c) Give an example to show the rectangular product between sets is not **commutative**. That is, find two sets A, B, such that $A \times B$ is not equal to $B \times A$.

(d) Choose any two sets A and B and compute their rectangular product as a set of ordered pairs.

(e) Describe a set of ordered pairs which is not the rectangular product of two sets.

(f) If a set A has n elements and a set B has m elements, find how many elements are in the set $A \times B$ by induction on n.

Definition 2.6. The **rectangular plane** is the rectangular product of $\mathbb{R}$ with $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}$, denoted by $\mathbb{R}^2$. Below is a drawing of the plane with relative positions of some of the ordered pairs. The arrows denote the pattern continues. Sometimes, ordered pairs as elements of $\mathbb{R}^2$ are called **points**. The **origin** is defined to be the point $(0, 0)$, denoted O.

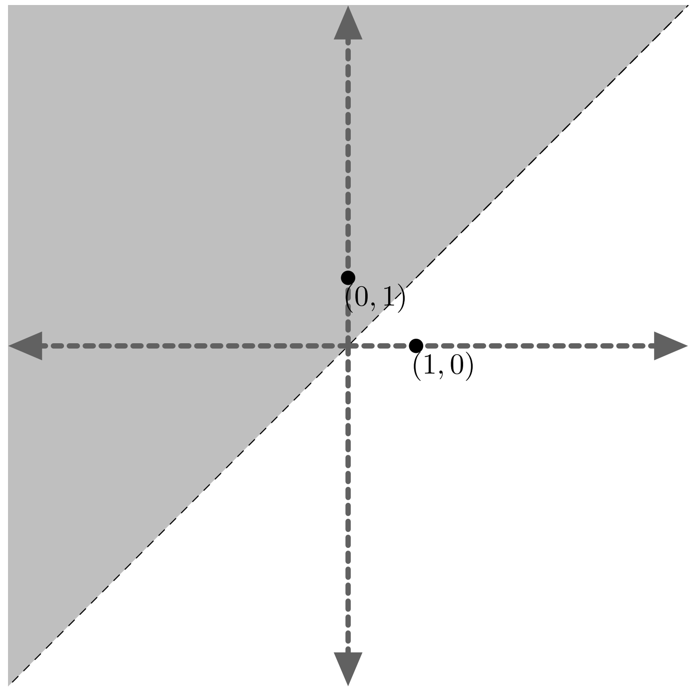


■

Example 2.7.

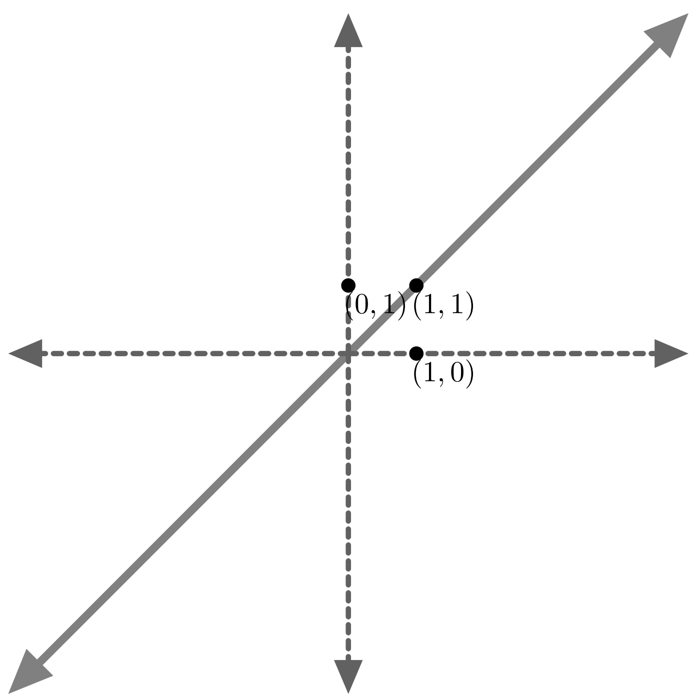
(a) The order relation $<$ between $\mathbb{R}$ and $\mathbb{R}$ can be interpreted as the set of all ordered pairs (a, b) of real numbers a and b for which $a < b$. In set-builder notation, the order relation on $\mathbb{R}$ is $\{(a, b) \in \mathbb{R}^2 \mid a < b\}$.

This relation consists of the shaded region below in the rectangular plane.



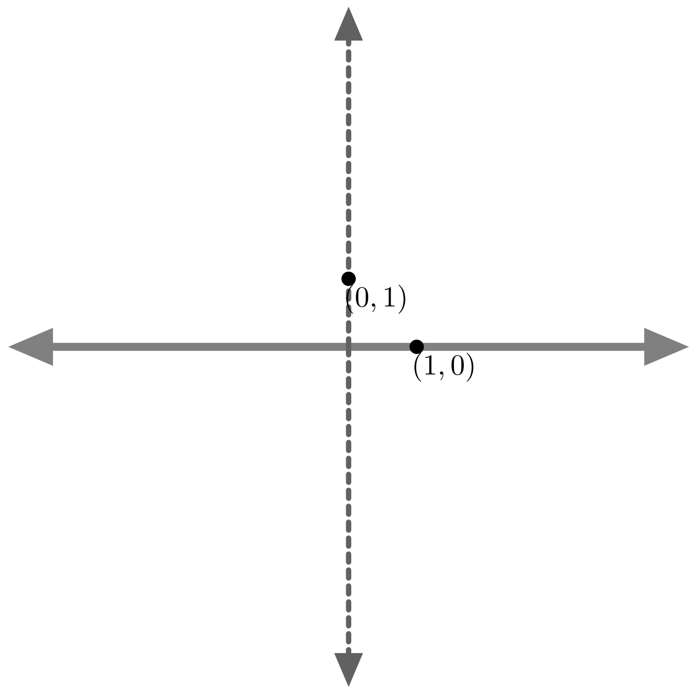
(b) Equality between real numbers is a relation between $\mathbb{R}$ and $\mathbb{R}$. In set-builder notation, equality between real numbers is $\{(a, b) \in \mathbb{R}^2 \mid a = b\}$.

The relation consists of the bold region in the rectangular plane.



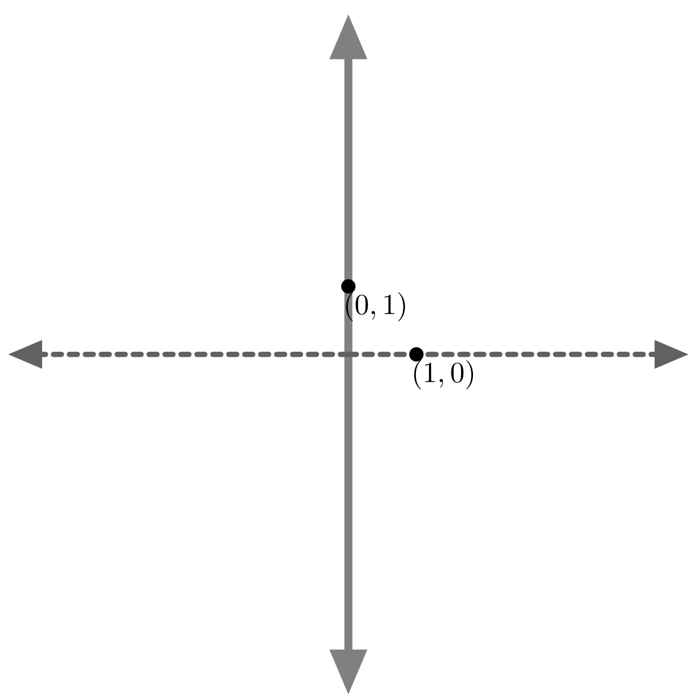
(c) The **horizontal axis** of $\mathbb{R}^2$ is the relation consisting of the points $(a, 0)$ for every real number a. In set-builder notation, the horizontal axis is $\{(a, b) \in \mathbb{R}^2 \mid b = 0\}$.

The relation consists of the bold region in the rectangular plane.



(d) The **vertical axis** of $\mathbb{R}^2$ is the relation consisting of the points $(0, b)$ for every real number b. In set-builder notation, the vertical axis is $\{(a, b) \in \mathbb{R}^2 \mid a = 0\}$.

The relation consists of the bold region in the rectangular plane.



■

2.2 Functions

Definition 2.8. Suppose A and B are sets. A **function** f with **domain** A and **codomain** B is a relation between A and B such that, for every a in A, there is some b in B for which $(a, b) \in f$. Moreover, for every (a, b) in f, if (a, c) is also in f, then $c = b$. Hence, if f is a function and (a, b) is in f, then we write $b = f(a)$ without ambiguity, and we call a the **input** of f and b its **output**. A function defined in this way is also called a **graph**. Hence, a function is a relation for which *every* element in its domain is related to some element in its codomain. Moreover, this relation is *unique*.

If f is a function with domain A and codomain B, its **range** is the set of all b in B for which $b = f(a)$ for some a in A. The, domain, codomain and range of a function f is denoted by domain(f), codomain(f), and range(f), respectively. In set-builder notation, if f is a function with domain A and codomain B,

$$\text{range}(f) = \{b \in B \mid (a, b) \in f \text{ for some } a \in A\}.$$

Another, equivalent way of writing this is

$$\text{range}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

■

Exercise 2.9. In Example 2.7, determine which are functions as relations in $\mathbb{R}^2$ and explain why.

Exercise 2.10.

For this exercise, let $B = \{a, b\}$, the set consisting of the characters a and b.

(a) If $A_1 = \{1\}$, the set consisting of the number 1, describe all functions with domain A_1 and codomain B.

(b) If $A_2 = \{1, 2\}$, describe all functions with domain A_2 and codomain B.

(c) If $A_3 = \{1, 2, 3\}$, describe all functions with domain A_3 and codomain B.

(d) If n is a natural number (an integer > 0), try to deduce a formula for the number of functions with domain consisting of n elements and codomain B. Can you explain your result?

Definition 2.11. If A is a set, there is a unique function, denoted id_A, with domain and codomain both A, which is the set of all ordered pairs (a, a) for every element a in A. Another way to specify this function is to say id_A is a function with domain and codomain A such that

$$\text{id}_A(a) \quad \text{if} \quad a \in A.$$

In general, this is how we can specify any function. If we know the domain and how each input is related to each output, then we've completely specified the function, as the next theorem demonstrates.

■

Theorem 2.12. If f and g are functions, then $f = g$ as relations if and only if

1. the domain of f is equal to the domain of g,

2. and $f(x) = g(x)$ for every x in the domain of f.

Proof. Suppose f and g are functions. Suppose the domain of f is A and its codomain is C and the domain of g is D and its codomain is B. First, suppose $f = g$ as relations. This means f as a subset of $A \times C$ is equal to g as a subset of $D \times B$. That is, (a, c) is in f exactly when (a, c) is in g.

1. We need to show $A = D$. If a is an element of A, then $(a, f(a))$ is in f. Hence, $(a, f(a))$ is in g. In particular, $(a, f(a))$ is in $D \times B$. Hence, a is an element of D. This shows A is a subset of D. If d is an element of D, then $(d, g(d))$ is in g. Hence, $(d, g(d))$ is in f. In particular, $(d, g(d))$ is in $A \times C$. Hence, d is an element of A. This shows D is a subset of A. Since A and D are subsets of each other, it follows $A = D$.

2. Since (a, c) is in f exactly when (a, c) is in g, it follows $f(a) = g(a)$ for all a in A.

Conversely, if items 1 and 2 hold, we need to show $f = g$ as relations. Suppose (a, c) is in f. Then a is in D, the domain of g by item 1, and $f(a) = c = g(a)$ by item 2. Hence, (a, c) is in g. Since $(a, c) \in f$ is arbitrary, this shows f is a subset of g. Similarly, g is a subset of f. Hence, $f = g$ as relations. □

We can't compare the codomains of two functions viewed as their graph. Two functions might be the same as relations but have different codomains. This is why we introduce the notion of a mapping.

Definition 2.13. A **mapping** is an ordered pair of a function's codomain and the function itself. If f is a function with codomain C, their mapping is given by (C, f). The **codomain of a mapping** is then its first coordinate, the **function of a mapping** is its second coordinate. The **domain** of a mapping is the domain of its function.

■

Theorem 2.14.

If (C, f) and (B, g) are mappings, then $(C, f) = (B, g)$ as mappings if and only if

1. the domain of f is equal to the domain of g,

2. the codomain of f is equal to the codomain of g,

3. and $f(x) = g(x)$ for every x in the domain of f.

Proof. This follows from the above theorem and Exercise 2.3 c. □

2.3 Unions, intersections and complements

Definition 2.15.

(a) The **union** of two sets A and B, denoted by $A \cup B$, is defined to be the set of all x in either A or B. That is, x is an element in $A \cup B$ if and only if either x is in A or x is in B.

(b) The **intersection** of two sets A and B, denoted by $A \cap B$, is defined to be the set of all x in both A and B. That is, x is an element in $A \cap B$ if and only if x is in A and x is in B. In set-builder notation,

$$A \cap B = \{x \in A \mid x \in B\}.$$

(c) Two sets are **disjoint** if their intersection equals the empty set. That is, two sets are disjoint if and only if they share no elements. In symbols, two sets A and B are disjoint if and only if $A \cap B = \emptyset$.

(d) If A is a subset of a set B, then the **complement** of A with respect to B is the set

$$\{x \in B \mid x \notin A\},$$

denoted $B \setminus A$.

■

Exercise 2.16. If A, B and C are sets,

(a) Explain why A and B are both subsets of $A \cup B$.

(b) Explain why $A \cap B$ is a subset of both A and B.

(c) Give an example of A and B with $A \cap B = A \cup B$.

(d) Give an example of A and B with $A \cap B \neq A \cup B$.

(e) Explain why $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(f) Explain why $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(g) Explain why, if A is a subset of B, then A and $B \setminus A$ are disjoint.

(h) Explain why, if A is a subset of B, then $A \cup (B \setminus A) = B$.

(i) If A and B are subsets of C, then $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$.

(j) If A and B are subsets of C, then $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.

(k) Explain why $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Exercise 2.17.

(a) Express the real number line as a disjoint union of two intervals.

(b) Express the real number line as a disjoint union of three intervals.

(c) Express the relation $\leq$ as a union of two relations.

Definition 2.18. If f and g are functions such that their union, $f \cup g$, as a relation with domain domain$(f) \cup$ domain(g) and codomain codomain$(f) \cup$ codomain(g), is a function, we write

$$f \cup g(x) = \begin{cases} f(x) & \text{if} \quad x \text{ is in domain}(f) \\ g(x) & \text{if} \quad x \text{ is in domain}(g) \end{cases}$$

to completely specify the piecewise function $f \cup g$.

■

Theorem 2.19. (Domain of union)
If f and g are functions, for which $f \cup g$ is a function, then

$$\text{range}(f \cup g) = \text{range}(f) \cup \text{range}(g).$$

Proof. If

$$y \in \text{range}(f \cup g)$$

if and only if there is some $x \in \text{domain}(f \cup g)$ for which

$$(x, y) \in f \cup g$$

if and only if either

$$(x, y) \in f \quad \text{or} \quad (x, y) \in g$$

if and only if either

$$y \in \text{range}(f) \quad \text{or} \quad y \in \text{range}(g).$$

□

Exercise 2.20. Determine when the union of two functions $f \cup g$ is itself a function. Such a function is called a **piecewise function**. Hint: If the domain of f and the domain of g are disjoint, then $f \cup g$ is a function.

2.4 composition

Definition 2.21. We define $f \circ g$, the **composition of** f **with** g, sometimes read f **of** g whose codomain is equal to the codomain of f, and for which

$$f \circ g(x) = f(g(x))$$

for all x in the domain of g for which $g(x)$ is in the domain of f.

■

Exercise 2.22. Describe two nonempty functions such that their composition is empty.

Theorem 2.23. (Composition of Piecewise) If f, g, l, m are functions for which $f \cup g$ and $l \cup m$ are functions, then

$$(f \cup g) \circ (l \cup m) = f \circ l \cup \ g \circ l \cup \ f \circ m \cup \ g \circ m$$

as relations. This is a type of distributive property.

Proof. We will apply the definitions of piecewise functions along with function composition and Theorem 2.12.

First, according to Theorem 2.12, we need to show the domains of either side are the same. By Definition 2.18 and the definition of a composition, it follows

$$x \in \text{domain}((f \cup g) \circ (l \cup m))$$

if and only if

$$x \in \text{domain}(l) \cup \text{domain}(m) \qquad \text{and} \qquad l \cup m(x) \in \text{domain}(f) \cup \text{domain}(g)$$

if and only if either

$$l(x) \in \text{domain}(f),$$
$$l(x) \in \text{domain}(g),$$
$$m(x) \in \text{domain}(f), \text{ or}$$
$$m(x) \in \text{domain}(g)$$

if and only if

$$x \in f \circ l \cup \ g \circ l \cup \ f \circ m \cup \ g \circ m.$$

We now need to show

$$(f \cup g) \circ (l \cup m)(x) = f \circ l \cup \ g \circ l \cup \ f \circ m \cup \ g \circ m(x)$$

for every x in this common domain. Then, by Theorem 2.12, we will have shown these two relations are equal.

If x is in the domain of l for which $l(x)$ is in the domain of f, then $x \in \text{domain}(f \circ l)$ and

$$(f \cup g) \circ (l \cup m)(x) = f \cup g(l(x)) = f(l(x)) = f \circ l(x).$$

Similarly, if x is in the domain of l for which $l(x)$ is in the domain of g, then $x \in \text{domain}(g \circ l)$ and

$$(f \cup g) \circ (l \cup m)(x) = f \cup g(l(x)) = g(l(x)) = g \circ l(x).$$

Similarly, if x is in the domain of m for which $m(x)$ is in the domain of f, then $x \in \text{domain}(f \circ m)$ and

$$(f \cup g) \circ (l \cup m)(x) = f \cup g(m(x)) = f(m(x)) = f \circ m(x).$$

Similarly, if x is in the domain of m for which $m(x)$ is in the domain of g, then $x \in \text{domain}(g \circ m)$ and

$$(f \cup g) \circ (l \cup m)(x) = f \cup g(m(x)) = g(m(x)) = g \circ m(x).$$

These are the only four possibilities for x to be in the domain of $(f \cup g) \circ (l \cup m)$. Hence,

$$(f \cup g) \circ (l \cup m)(x) = f \circ l \cup \; g \circ l \cup \; f \circ m \cup \; g \circ m(x)$$

for every x in this common domain. □

2.4.1 image and preimage

Sometimes we'd like to find the inputs of a function which evaluate to zero. All such inputs form what we call a preimage of the set consisting of the number zero. The domain of $f \circ g$ is another example of a preimage. A preimage of a function is a subset of its domain. An image of a function is a subset of its codomain.

Definition 2.24.

(a) If S is a set and f is a function, then the **preimage of S under f** is the set

$$\{x \in \text{domain}(f) \mid f(x) \in S\}$$

in set-builder notation. We denote the preimage of S under f by $f^{-1}(S)$.

(b) If D is a set and f is a function, then the **image of D under f** is the set

$$\{y \in \text{codomain}(f) \mid y = f(x) \text{ for some } x \in D\}.$$

We denote the image of D under f by $f(D)$.

■

Exercise 2.25. Suppose f and g are both functions.

(a) Determine exactly when the preimage of a set under f is empty.

(b) Determine exactly when the image of a set under f is empty.

(c) Show, if A and B are both subsets of the codomain of f, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(d) Show, if A and B are both subsets of the codomain of f, then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(e) Show, if A and B are both subsets of the domain of f, then $f(A \cup B) = f(A) \cup f(B)$.

(f) Show, if A and B are both subsets of the domain of f, then $f(A \cap B) = f(A) \cap f(B)$.

(g) Explain why, if C is a subset of the codomain of f and A is a subset of C, then $f^{-1}(C \setminus A) = f^{-1}(C) \setminus f^{-1}(A)$.

(h) Explain why, if C is a subset of the domain of f and A is a subset of C, then $f(C) \setminus f(A)$ is a subset of $f(C \setminus A)$.

(i) Give an example of a function f, a subset of its domain C and a subset A of C for which $f(C \setminus A)$ is not a subset of $f(C) \setminus f(A)$.

(j) Explain why $f^{-1}(\text{codomain}(f)) = \text{domain}(f)$.

(k) Explain why $f(\text{domain}(f)) = \text{range}(f)$.

(l) If A is a subset of the domain of f, explain why $f^{-1}(f(A)) = A$.

(m) If A is a subset of the range of f, explain why $f(f^{-1}(A)) = A$.

(n) If A is a subset of the codomain of f, explain why $f(f^{-1}(A))$ is a subset of A.

(o) Give an example of a function f and subset of its codomain A for which A is not a subset of $f(f^{-1}(A))$.

(p) Find the domain of $f \circ g$ in terms of the domain of f and g as a preimage.

(q) Show, if A is any subset of the codomain of g, then $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$.

(r) Show, if A is any subset of the domain of g, then $(f \circ g)(A) = f(g(A))$.

Exercise 2.26. If f, g, l, m are functions for which $f \cup g$ and $l \cup m$ are functions, find the domain of $(f \cup g) \circ (l \cup m)$ as a union of preimages.

2.5 Function restriction

Definition 2.27.

(a) If f is a function and A a subset of its domain, we define the new function $f|_A$, called the **restriction of** f **to** A, whose domain is A, codomain is codomain(f) and for which

$$f|_A(x) = f(x) \quad \text{if} \quad x \in A.$$

If B is a set and A is a subset of B, then **inclusion function** $\iota_{A,B}$ from A into B is

$$\iota_{A,B} = \text{id}_B|_A.$$

ι is the Greek letter iota.

(b) If f is a function and B a subset of its codomain the **right-restriction of** f **to** B is the mapping (B, f).

■

Exercise 2.28.

(a) If f is a function, then

$$f|_{\text{domain}(f)} = f.$$

(b) If f is a function, S a subset of its codomain and A a subset of its domain, then

$$(f|_A)^{-1}(S) = f^{-1}(S) \cap A.$$

(c) If f and g are functions and A is a subset of the domain of g, then explain why

$$(f \circ g)|_A = f \circ (g|_A).$$

2.6 Real functions

Definition 2.29. A **real-valued** function is a function whose codomain is a subset of the real numbers. A **real function** is a real-valued function whose domain is a subset of the real line.

Definition 2.30.

(a) If A is a set and r is a real number, the real-valued r**-constant function on** A, denoted $\text{const}_{A,r}$, with domain A and codomain $\{r\}$ is defined by

$$\text{const}_{A,r}(a) = r \quad \text{if} \quad a \in A.$$

The r-constant function on A outputs the number r for every input in A. If $A = \mathbb{R}$, we write $\text{const}_{\mathbb{R},r} = \text{const}_r$ for simplicity.

(b) If B is a set and A is a subset of B, the real-valued **indicator function of** A **of** B is the real-valued function, denoted $\chi_{A,B}$, with domain B and codomain $\{0, 1\}$, defined by

$$\chi_{A,B} = \text{const}_{A,1} \cup \text{const}_{B \setminus A,0}.$$

The indicator of A as a subset of B tells us whether an element is in A or not. χ is the Greek letter chi.

Exercise 2.31. If a is any real number, and f is any function, explain either why or why not

(a) $\text{const}_a \circ f = \text{const}_a$ as functions.

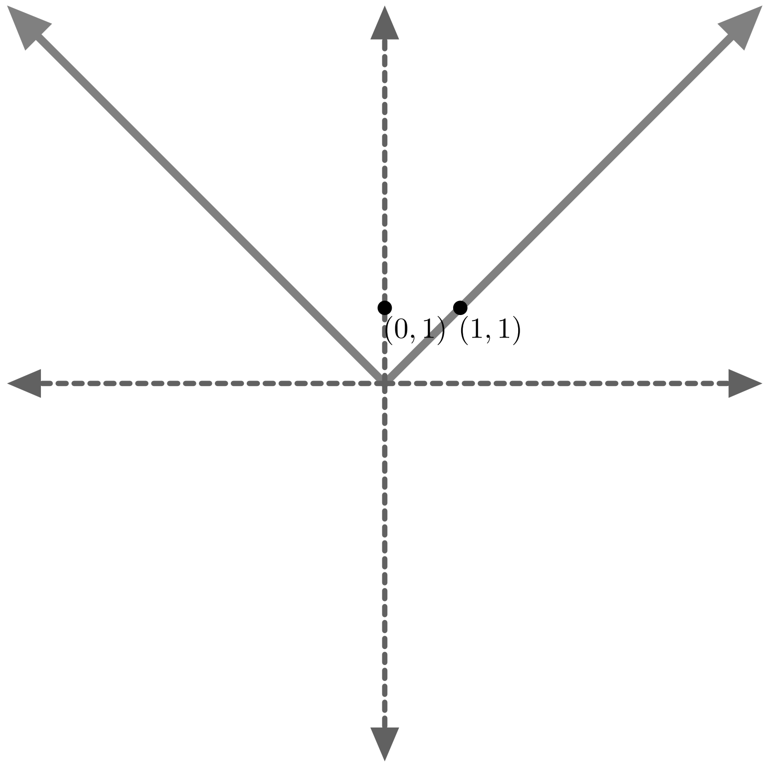
(b) $f \circ \ \text{const}_a = \text{const}_{f(a)}$ as functions.

Example 2.32. The **absolute value function**, abs, with domain $\mathbb{R}$ and codomain $\mathbb{R}$, is the function consisting of the ordered pairs $(a, |a|)$ for every real number a. In other words,

$$\text{abs}(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

So, $|x|$ and $\text{abs}(x)$ are two different notations for the same number.

This relation consists of the bold region in the rectangular plane.



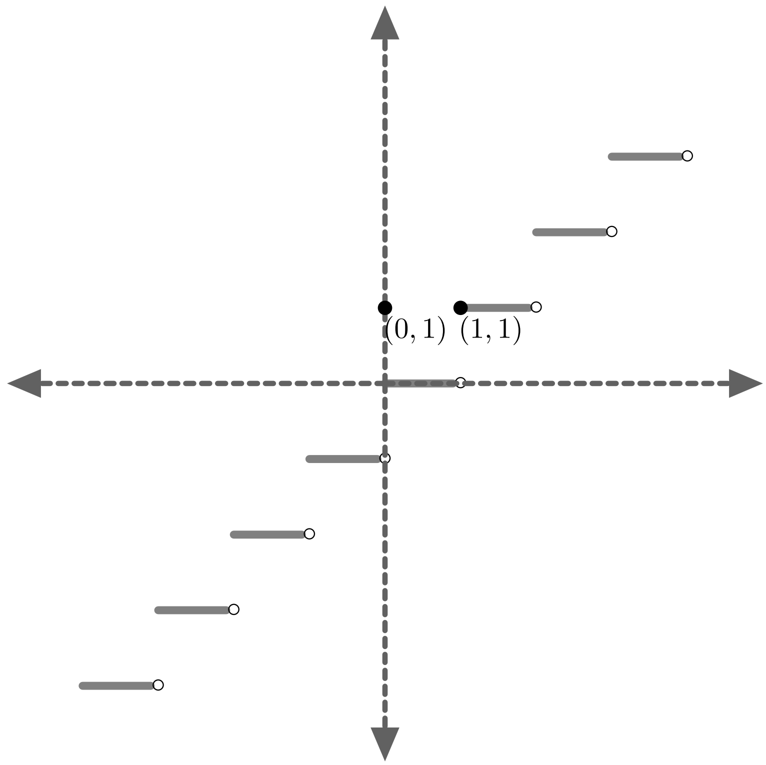
∎

Example 2.33. We define the **floor** function by rounding each real number down to the nearest integer. For any real number x, there is a unique integer n for which $n \leq x < n+1$. We denote such n floor(x). This can be stylized as a piecewise function as

$$\text{floor}(x) = \{n \quad \text{if} \quad n \leq x < n+1.$$

We can also write

$$\text{floor} = \bigcup_{n>-\infty}^{\infty} \text{const}_n|_{[n,n+1[}.$$



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Exercise 2.34.

1. Find the range of abs. Express this set as an interval.
2. Find the range of floor.

Definition 2.35.

(a) If A and B are subsets of the real numbers, define their **sum** $A+B := \{x+y \mid x \in A \text{ and } y \in B\}$.

(b) If f and g are real functions, define their **sum** $f + g$ with domain $\text{domain}(f) \cap \text{domain}(g)$, codomain $\text{codomain}(f) + \text{codomain}(g)$, for which $(f + g)(x) = f(x) + g(x)$ for all x in the intersection of the domain of f with the domain of g.

(c) If A and B are subsets of the real numbers, define their **product** $A \cdot B := \{xy \mid x \in A \text{ and } y \in B\}$.

(d) If f and g are real functions, define their **product** $f \cdot g$ with domain $\text{domain}(f) \cap \text{domain}(g)$, codomain $\text{codomain}(f) \cdot \text{codomain}(g)$, for which $(f \cdot g)(x) = f(x) \cdot g(x)$ for all x in the intersection of the domain of f with the domain of g.

The conventional order of operations still holds for addition and multiplication of functions. So that $f + g \cdot h$ means $f + (g \cdot h)$ and we have the distributive property $f \cdot (g + h) = f \cdot g + f \cdot h$.

■

Exercise 2.36.

(a) Define the **sign function**, sgn, with domain $\mathbb{R}$ and codomain $\{-1, 0, 1\}$ by

$$\operatorname{sgn}(t) = \begin{cases} -1 & \text{if} \quad t < 0 \\ 0 & \text{if} \quad t = 0 \\ 1 & \text{if} \quad t > 0. \end{cases}$$

Show $\operatorname{id}_{\mathbb{R}} = \operatorname{sgn} \cdot \operatorname{abs}$.

(b) The **difference** of two real-valued functions f and g is defined by

$$f - g := f + \operatorname{const}_{-1} \cdot g.$$

That is, for all $x \in \operatorname{domain}(f) \cap \operatorname{domain}(g)$, $f - g(x) = f(x) - g(x)$. Express sgn as the the difference of two indicator functions.

(c) Express the absolute value function as the union of functions involving the identity on certain intervals and multiplication by a constant.

(d) Express the sign function as the union of functions involving the identity on certain intervals and multiplication by a constant.

Definition 2.37. Polynomials

(a) We say const_0 is the unique **polynomial of degree** $-\infty$ (minus infinity). const_0 is sometimes referred to as the **zero polynomial**.

(b) If n is a nonnegative integer we define the **monic monomial of degree** n, denoted $\operatorname{id}_{\mathbb{R}}^n$ by the following recursive formula.

  (a) $\operatorname{id}_{\mathbb{R}}^n := \operatorname{const}_1$ if $n = 0$,
  (b) $\operatorname{id}_{\mathbb{R}}^n := \operatorname{id}_{\mathbb{R}} \cdot \operatorname{id}_{\mathbb{R}}^{n-1}$ if $n \in \mathbb{N}$.

That is, if n is a nonnegative integer, the monic monomial of degree n is a function defined on $\mathbb{R}$ for which $\operatorname{id}_{\mathbb{R}}^n(x) = x^n$ for all real numbers x. Recall Definition A.9.

(c) If n is a nonnegative integer and a is a nonzero real number, the **monomial of degree** n **and coefficient** a is the function

$$\operatorname{const}_a \cdot \operatorname{id}_{\mathbb{R}}^n.$$

Note the monic monomial of degree n is the monomial of degree n and coefficient 1.

(d) If n is a nonnegative integer and a is a nonzero real number, we say the real function p is a **polynomial of degree** n **and leading coefficient** a, if and only if $p = \operatorname{const}_a \cdot \operatorname{id}_{\mathbb{R}}^n + q$ for some $n \in \mathbb{N}$ and some polynomial q of degree less than n.

  - Note q can be the zero polynomial by this definition. So, every monomial is a polynomial.

- In addition, this recursive definition implies, for every polynomial of degree n, p, there are unique real numbers $a_0, \dots, a_n$ for which a_n is not zero, such that

$$p = \sum_{i=0}^{n} \mathrm{const}_{a_i} \cdot \mathrm{id}_{\mathbb{R}}^{i}.$$

Remember this is **summation notation** first introduced in these notes in Definition A.6. The only difference is the summands (the terms we are summing) in this case are functions rather than numbers. That is,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

if x is real. Here, the summands are numbers. $a_0, \dots, a_n$ are the **coefficients** of p. a_0 is its **constant coefficient**. a_0 is the output of the polynomial function with input 0.

(e) An **affine function** is a polynomial of degree less than or equal to 1. A **linear function** is an affine function which is either constant or whose constant coefficient is zero. Most Precalculus or college algebra books will call what we call an affine function a linear function and not make a distinction between the two. Thus, an affine function is determined by two coefficients. We introduce the following notations for affine functions. If a and b are real numbers, the affine function $l_{a,b}$ is defined as

$$l_{a,b} = \mathrm{const}_a \cdot \mathrm{id}_{\mathbb{R}} + \mathrm{const}_b.$$

Thus, $l_{a.b}(x) = ax + b$ if x is real. Hence, a linear function is of the form $l_{0,b}$ with b not zero or $l_{a,0}$ with a not zero. $l_{0,b}(x) = b$ for every real x, a constant. $l_{a,0}(x) = ax$ for every real x, **multiplication on the left**. If b is real, $l_{1,b}$ is **addition on the right**. $l_{1,b}(x) = x + b$. This is affine and not linear unless $b = 0$, in which case it's the identity on $\mathbb{R}$. As the degree of the zero polynomial is $-\infty$ by convention, it's also an affine function. In fact, the zero polynomial is linear. Denoted by const_0 or $l_{0,0}$.

(f) A **quadratic function** is a polynomial of degree 2. Notationally, if a, b, c are real numbers and a is not zero, then we denote the quadratic function $\mathrm{const}_a \cdot \mathrm{id}_{\mathbb{R}}^2 + \mathrm{const}_b \cdot \mathrm{id}_{\mathbb{R}} + \mathrm{const}_c$ as

$$q_{a,b,c}.$$

That is,

$$q_{a,b,c}(x) = ax^2 + bx + c$$

if x is real. Note the notation $q_{a,b,c}$ is reserved for degree 2 polynomials, and so is only valid if $a \neq 0$.

■

Exercise 2.38.

(a) Demonstrate that a polynomial is completely determined by its coefficients. That is, if n and m are nonnegative integers and $a_0, \dots, a_n, b_0, \dots, b_m$ are real numbers such that $a_n, b_m \neq 0$ and

$$\sum_{i=0}^{n} \mathrm{const}_{a_i} \cdot \mathrm{id}_{\mathbb{R}}^{i} = \sum_{i=0}^{m} \mathrm{const}_{b_i} \cdot \mathrm{id}_{\mathbb{R}}^{i}$$

as relations, then $n = m$ and $a_i = b_i$ for all $i = 0, \dots, n$. *Hint:* Induction on n starting at 0.

(b) If m_1, m_2, b_1, b_2 are real numbers, express $l_{m_2,b_2} \circ l_{m_1,b_1}$ as an affine function.

(c) If m_1, m_2, b_1, b_2 are real numbers, express $l_{m_2,b_2} + l_{m_1,b_1}$ as an affine function.

(d) If m_1, m_2, b_1, b_2 are real numbers, express $l_{m_2,b_2} \cdot l_{m_1,b_1}$ as a quadratic function.

(e) Demonstrate, if n and m are $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are real numbers, then

$$\left(\sum_{i=0}^{n} \mathrm{const}_{a_i} \cdot \mathrm{id}_{\mathbb{R}}^{i}\right) \cdot \left(\sum_{i=0}^{m} \mathrm{const}_{b_j} \cdot \mathrm{id}_{\mathbb{R}}^{j}\right) = \sum_{k=0}^{n+m} \left(\sum_{i=0}^{k} \mathrm{const}_{a_i b_{k-i}}\right) \cdot \mathrm{id}_{\mathbb{R}}^{k}$$

by induction on n. This shows the degree of the product of two polynomials is the sum of their degrees.

(f) Demonstrate by way of example. The degree of the sum of two polynomials can be equal to the maximum of their degrees.

(g) Demonstrate by way of example. The degree of the sum of two polynomials can be less than the maximum of their degrees.

(h) Provide an example of two degree three polynomials whose sum has degree 0.

(i) What is the degree of the composition of two polynomials in terms of their degrees?

2.6.1 Zeroth and First order data

Definition 2.39. Zeroth order data

For the following definitions, we suppose f is real-valued function and A is a subset of its domain.

- f is **positive on** A if $f(x) > 0$ whenever $x \in A$. That is, f is positive on A if A is a subset of $f^{-1}((0, \infty))$.
- f is **negative on** A if $f(x) < 0$ whenever $x \in A$. That is, f is negative on A if A is a subset of $f^{-1}((-\infty, 0))$.
- An element of the domain of f, c, is a **zero of** f if $f(c) = 0$. That is, c is a zero of f if $c \in f^{-1}(\{0\})$.

■

Definition 2.40. First order data

For the following definitions, we suppose f is real function, and I is any interval, bounded or unbounded, open, closed, or half.

- f is **constant on** I if $f(x_1) = f(x_2)$ whenever $x_1 \in I$ and $x_2 \in I$.
- f is **strictly increasing on** I if, whenever $x_1 \in I$, $x_2 \in I$ and $x_1 < x_2$, then

$$f(x_1) < f(x_2)$$

and **increasing** if $f(x_1) \leq f(x_2)$ instead.

- f is **strictly decreasing on** I if, whenever $x_1 \in I$, $x_2 \in I$ and $x_1 < x_2$, then

$$f(x_2) < f(x_1)$$

and **decreasing** if $f(x_2) \leq f(x_1)$ instead.

■

Definition 2.41. If a, b are numbers in the domain of a real function f, the **average rate of change** of f between a and b is the real number

$$\frac{f(b) - f(a)}{b - a}.$$

■

The average rate of change of a function between two numbers is an approximate measure of how much the function is increasing or decreasing on the closed interval with those numbers as endpoints.

Exercise 2.42. If f is a real function defined on an interval I, explain the following.

(a) f is strictly increasing on I if and only if the average rate of change of f between any two members of I is positive.

(b) f is strictly decreasing on I if and only if the average rate of change of f between any two members of I is negative.

(c) f is constant on I if and only if the average rate of change of f between any two members of I is zero.

Exercise 2.43.

(a) If $h \neq 0$, find the average rate of change of abs between 0 and h. Express your result as a piecewise function of h.

(b) On which intervals is abs strictly increasing? Where is it strictly decreasing?

Theorem 2.44. If $m > 0$ and d is a real number, then

1. $l_{m,d}^{-1}((0, \infty)) = (-\frac{d}{m}, \infty)$. That is, the affine function $l_{m,d}$ is positive on the interval $(-\frac{d}{m}, \infty)$ if $m > 0$.

2. $l_{m,d}^{-1}((-\infty, 0)) = (-\infty, -\frac{d}{m})$.

3. $l_{m,d}^{-1}(\{0\}) = \{-\frac{d}{m}\}$. That is, the only zero of $l_{m,d}$ is $-\frac{d}{m}$.

4. $l_{m,d}$ is strictly increasing on $\mathbb{R}$.

5. The average rate of change of $l_{m,d}$ is m on every bounded closed interval of $\mathbb{R}$.

Proof. Throughout, we employ Theorem 1.13.

1. To show $l_{m,d}^{-1}((0,\infty)) = (-\frac{d}{m}, \infty)$, we need to show they are subsets of each other.

   - If $x \in l_{m,d}^{-1}((0,\infty))$, then $l_{m,d}(x) > 0$. Hence, $mx + d > 0$. Hence, $mx > -d$. Hence, since $m > 0$, $x > -\frac{d}{m}$. Hence, $x \in (-\frac{d}{m}, \infty)$. Since x is an arbitrary element of $l_{m,d}^{-1}((0,\infty))$, this demonstrates $l_{m,d}^{-1}((0,\infty))$ is a subset of $(-\frac{d}{m}, \infty)$.
   - If $x \in (-\frac{d}{m}, \infty)$, then $x > -\frac{d}{m}$. Since $m > 0$, it follows $mx > -d$. Hence, $mx + d > 0$. Hence, $l_{m,d}(x) > 0$. Hence, $x \in l_{m,d}^{-1}((0,\infty))$. Since x is an arbitrary element of $(-\frac{d}{m}, \infty)$, this demonstrates $(-\frac{d}{m}, \infty)$ is a subset of $l_{m,d}^{-1}((0,\infty))$.

   Hence, these two sets are subsets of each other. So, they are equal.

2. This is can be copied almost verbatim from (a).

3. We can demonstrate $l_{m,d}^{-1}(\{0\}) = \{-\frac{d}{m}\}$ in a similar way we demonstrated parts (a) and (b). We can also demonstrate this part in the following way. Since part (a) and (b) are proved, the codomain and domain of $l_{m,d}$ is $\mathbb{R}$, $\{0\} = \mathbb{R} \setminus ((-\infty, 0) \cup (0, \infty))$, then by Exercise 2.25, c, g and j,

$$\begin{aligned}
l_{m,d}^{-1}(\{0\}) &= l_{m,d}^{-1}(\mathbb{R} \setminus ((-\infty, 0) \cup (0, \infty))) \\
&= l_{m,d}^{-1}(\mathbb{R}) \setminus l_{m,d}^{-1}((-\infty, 0) \cup (0, \infty)) \\
&= \mathbb{R} \setminus (l_{m,d}^{-1}((-\infty, 0)) \cup l_{m,d}^{-1}((0, \infty))) \\
&= \mathbb{R} \setminus \left(\left(-\infty, -\frac{d}{m} \right) \cup \left(-\frac{d}{m}, \infty \right) \right) \\
&= \left\{ -\frac{d}{m} \right\}.
\end{aligned}$$

4. If x_1 and x_2 are real numbers and $x_1 < x_2$, then, since $m > 0$, it follows $mx_1 < mx_2$ because multiplication by a positive constant preserves order. Hence, $mx_1 + d < mx_2 + d$. Hence, $l_{m,d}(x_1) < l_{m,d}(x_2)$. Since x_1 and x_2 are arbitrary real numbers, it follows $l_{m,d}$ is strictly increasing on $(-\infty, \infty)$.

5. If a and b are real numbers and $a < b$, then since the domain of $l_{m,d}$ is $\mathbb{R}$, a and b are in the domain of $l_{m,d}$. And

$$\begin{aligned}
\frac{l_{m,d}(b) - l_{m,d}(a)}{b - a} &= \frac{mb + d - (ma + d)}{b - a} \\
&= \frac{mb + d - ma - d}{b - a} \\
&= \frac{mb - ma + d - d}{b - a} \\
&= \frac{mb - ma}{b - a} \\
&= m \cdot \frac{b - a}{b - a} \\
&= m.
\end{aligned}$$

Since a and b are arbitrary real numbers, it follows the average rate of change of $l_{m,d}$ is m between any two real numbers.

□

Exercise 2.45.

(a) Deduce similar facts when $m < 0$.

(b) A function is linear exactly when its average rate of change between any two numbers is the same constant (depending on the function).

(c) If m_1 and m_2 are real numbers, is $l_{m_1,0} \circ l_{m_2,0}$ strictly increasing, strictly decreasing, constant, or none of the above? Why?

Theorem 2.46.

- The composition of either two increasing functions or two decreasing functions is increasing. In particular, if f is increasing on (a, b), and g is increasing on (c, d), then $f \circ g$ is increasing on $g^{-1}((a, b)) \cap (c, d)$.
- The composition of an increasing function and a decreasing function is decreasing.

Proof.

- With this set up, suppose $x_1 < x_2 \in g^{-1}((a, b)) \cap (c, d)$. Then $g(x_1) \leq g(x_2) \in (a, b)$ since g is increasing on (c, d). Hence, since f is increasing on (a, b), $f \circ g(x_1) = f(g(x_1)) \leq f(g(x_2)) = f \circ g(x_2)$. Hence, $f \circ g$ is increasing on $g^{-1}((a, b)) \cap (c, d)$.
- If f is increasing on (a, b), and g is decreasing on (c, d), then consider $x_1 < x_2 \in g^{-1}((a, b)) \cap (c, d)$. Then $g(x_2) \leq g(x_1)$. Hence, $f \circ g(x_2) = f(g(x_2)) \leq f(g(x_1)) = f \circ g(x_1)$. Hence, $f \circ g$ is decreasing on $g^{-1}((a, b)) \cap (c, d)$. $f \circ g$ is also decreasing if f is decreasing and g is increasing.

□

Exercise 2.47. Suppose f and g are real functions. Explain the following claims.

(a) If f and g are both increasing on a set I, then $f + g$ is increasing on I.

(b) If f is increasing on a set I and g is positive on I, then $f \cdot g$ is increasing on I.

(c) If f is increasing on a set I and g is negative on I, then $f \cdot g$ is decreasing on I.

(d) There are increasing functions f and g on an interval I for which $f \cdot g$ is neither increasing nor decreasing on I.

(e) There is an increasing function f and a positive function g on an interval I for which $f + g$ is neither increasing nor decreasing on I.

Definition 2.48. If c is in the domain of a real-valued function f, then $f(c)$ is a **minimum value** if $f(c) \leq f(x)$ for all $x \in \text{domain}(f)$. A real function f has a **local minimum value** $f(c)$ if there exists an open interval (a, b) containing c such that $f(c)$ is a minimum value of $f|_{(a,b)}$.

If c is in the domain of a real-valued function f, then $f(c)$ is a **maximum value** if $f(c) \geq f(x)$ for all $x \in \text{domain}(f)$. A real function f has a **local maximum value** $f(c)$ if there exists an open interval (a, b) containing c such that $f(c)$ is a maximum value of $f|_{(a,b)}$.

A maximum or minimum value is referred to as an **extreme value**.

■

Exercise 2.49. Explain the following claims.

(a) If a real function is strictly increasing or strictly decreasing on an open interval, then it has no maxima nor minima in that interval.

(b) There are numbers $a < b < c$ and a function s, strictly decreasing on (a, b), strictly increasing on (b, c) for which s does not have a local minimum at c.

(c) There are numbers $a < b < c$ and a function s, strictly decreasing on (a, b), strictly increasing on (b, c) for which s has a local maximum at c.

(d) There is an example of a function which is decreasing on the interval $(-1, 1)$ but not on $[-1, 1)$.

(e) There is a function which has two local minimum values but no local maximum values.

(f) There is a function defined on $\mathbb{R}$ which is neither strictly increasing nor strictly decreasing on any interval.

Theorem 2.50. For a real function g, if $g(x_1)$ is a local minimum value of g at x_1, and if f is a real function, increasing on its domain, and $g(x_1) \in \text{domain}(f)$, then $f(g(x_1))$ is a local minimum value of $f \circ g$ at x_1.

Proof. If g is a real function with a local minimum value $g(x_1)$, then there is some real a, b for which $x_1 \in \text{domain}(g) \cap (a, b)$ and $g(x) \geq g(x_1)$ for all $x \in \text{domain}(g) \cap (a, b)$. Hence, if $x \in \text{domain}(f \circ g) \cap (a, b) = g^{-1}(\text{domain}(f)) \cap (a, b)$, then $f(x) \geq f(g(x_1))$ since f is increasing on its domain and $g^{-1}(\text{domain}(f)) \cap (a, b)$ is a subset of its domain. Since $x_1 \in g^{-1}(\text{domain}(f)) \cap (a, b)$, it follows $f(g(x_1))$ is a local minimum value of $f \circ g$ at x_1. □

In this theorem, we are **post composing** a function, whose minimum is of interest, with an increasing function. We are changing the range of g, not the domain, and to preserve the extreme value we must preserve the order of the range. If f is decreasing and g has a local minimum, then $f \circ g$ has a local maximum.

We also have

Theorem 2.51. For a real function f, if $f(y_1)$ is a local minimum value of f at y_1, and if g is any real function for which y_1 is in range of g, say $g(x_1) = y_1$ for some $x_1 \in \text{domain}(g)$, then $f(g(x_1))$ is a local minimum value of $f \circ g$ at x_1.

Proof. There is some interval (c,d) for which $y_1 \in \text{domain}(f) \cap \in (c,d)$ and $f(y) \geq f(y_1)$ for all $y \in \text{domain}(f) \cap (c,d)$. Then by Exercise 2.25, $g^{-1}(\text{domain}(f) \cap (c,d)) = g^{-1}(\text{domain}(f)) \cap g^{-1}((c,d)) = \text{domain}(f \circ g) \cap g^{-1}((c,d))$. Then let b be the least upper bound of $g^{-1}((c,d))$ and a the least upper bound of $\text{const}_{-1}(g^{-1}((c,d)))$. Then $g^{-1}(\text{domain}(f) \cap (c,d))$ is a subset of $\text{domain}(f \circ g) \cap (a,b)$. Hence, $x_1 \in \text{domain}(f \circ g) \cap (a,b)$ and $f(g(x_1)) \leq f(g(x))$ if $x \in \text{domain}(f \circ g) \cap (a,b)$. □

Notice in this theorem, we are **pre composing** a function, whose minimum is of interest, with any real function. We are changing the domain of f and hence where the local extreme value occurs.

2.6.2 global data

Definition 2.52. Suppose f is a real function. We say

(a)

$$\lim_{-\infty} f = \infty$$

to mean the following sentence.

For any real number M, there is some real number m for which the image $f((-\infty, m))$ is a nonempty subset of the unbounded open interval (M, ∞).

The nonempty condition ensures the trivial situation where f is not defined on an unbounded interval is excluded.

$$\lim_{-\infty} f = \infty$$

is read f approaches ∞ at $-\infty$ or the limit of f at $-\infty$ is ∞. Roughly, this means we can make $f(x)$ positive and as large as we wish given we make x negative and $|x|$ sufficiently large.

(b) We say

$$\lim_{\infty} f = \infty$$

to mean the following sentence.

For any real number M, there is some real number N for which the image $f((N, \infty))$ is a nonempty subset of the unbounded open interval (M, ∞).

(c) We also have the definitions

$$\lim_{-\infty} f = -\infty$$

and

(d)

$$\lim_{\infty} f = -\infty$$

If a real function f has any one of these four properties, we say f has an **infinite limit at infinity**.

If b is a real number, we say

(a)
$$\lim_{\infty} f = b$$
to mean for any positive real number r, there is some real number N for which $f((N, \infty))$ is a nonempty subset of $I_r(b)$. Roughly, we can make $f(x)$ as close to b as we wish given we make x sufficiently large.

(b) We can similarly define
$$\lim_{-\infty} f = b$$

In either one of these cases, we say f has a **finite limit at infinity**.

In the textbooks, sometimes $\lim_{-\infty} f = \infty$ is written $f(x) \to \infty$ as $x \to -\infty$, and so on.

■

Theorem 2.53. If $m > 0$, and d is real, the affine function $l_{m,d}$ has the following global data.

$$\lim_{\infty} l_{m,d} = \infty$$

and

$$\lim_{-\infty} l_{m,d} = -\infty.$$

Proof. Suppose $m > 0$ and d is real. Let's first demonstrate

$$\lim_{\infty} l_{m,d} = \infty.$$

by definition. Let M be an arbitrary real number. We wish to find a real number N for which $mx + d > M$ as long as $x > N$. $mx + d > M$ if and only if

$$x > \frac{M - d}{m}$$

since $m > 0$. So, if we define $N = \frac{M-d}{m}$, then, if $x > N$, then $l_{m,d}(x) = mx + d > M$. Hence, we've proved $l_{m,d}((\frac{M-d}{m}, \infty))$ is a nonempty subset of (M, ∞). Since M is arbitrary, this demonstrates

$$\lim_{\infty} l_{m,d} = \infty.$$

Let's now demonstrate

$$\lim_{-\infty} l_{m,d} = -\infty.$$

by definition. Let M be an arbitrary real number. We wish to find a real number N for which $mx + d < M$ as long as $x < N$. $mx + d < M$ if and only if

$$x < \frac{M - d}{m}$$

since $m > 0$. So, if we define $N = \frac{M-d}{m}$, then, if $x < N$, then $mx + d < M$. This demonstrates $l_{m,d}((-\infty, \frac{M-d}{m}))$ is a nonempty subset of $(-\infty, M)$. Since M is arbitrary, this demonstrates

$$\lim_{-\infty} l_{m,d} = -\infty.$$

□

Exercise 2.54.

(a) What are similar deductions when $m < 0$?

(b) If b is a real number,

$$\lim_{\infty} \text{const}_b = b$$

and

$$\lim_{-\infty} \text{const}_b = b.$$

---

3 Lines and linear systems

Definition 3.1.

(a) Given three real numbers a, b, c, the **line** determined by a, b, c is the following relation of the rectangular plane.
$$\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}.$$
Thus, a line is the set of all points (x, y) satisfying the equation $ax + by = c$ for some real numbers a, b, c.

(b) The line
$$\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$$
is **vertical** if $b = 0$ and $a \neq 0$.

(c) The line
$$\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$$
is **horizontal** if $a = 0$ and $b \neq 0$.

(d) The line
$$\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$$
is **trivial** if $a = b = 0$. In the case $a = b = c = 0$, this line is the entire plane. If $a = b = 0$ and $c \neq 0$, this line is the empty set.

■

Thus, a line is completely determined by an equation of the form $ax + by = c$ for some a, b, c real whose solutions (x, y) form the line itself.

Theorem 3.2.

(a) If L is a nontrivial vertical line, then there is some real number k for which
$$L = \{(k, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}.$$

(b) A nontrivial nonvertical line can be written as
$$\{(x, mx + d) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$
for some real numbers m and d. Thus, since
$$l_{m,d} = \{(x, mx + d) \in \mathbb{R}^2 \mid x \in \mathbb{R}\},$$
a nontrivial nonvertical line is an affine function.

Proof.

(a) Let L be a line. Then there are some real numbers a, b, c for which

$$L = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}.$$

If L is a vertical line, then $b = 0$ and $a \neq 0$. Hence, $(x, y) \in L$ if and only if $ax + by = c$ if and only if $ax = c$ if and only if $x = \frac{c}{a}$. So, if $k = \frac{c}{a}$, then

$$L = \{(k, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}.$$

(b) Let L be a line. Then there are some real numbers a, b, c for which

$$L = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}.$$

If L is nontrivial, we can't have both $b = 0$ and $a = 0$, since in case $c = 0$ as well, $L = \mathbb{R}^2$. In case $a = b = 0$ and $c \neq 0$, $L = \emptyset$. If L is nonvertical, then either $b \neq 0$ or $a = 0$. Hence, if L is nontrivial and nonvertical, $b \neq 0$ necessarily. If $b \neq 0$, then $(x, y) \in L$ if and only if $ax + by = c$ if and only if $y = \frac{c}{b} - \frac{a}{b}x$. Hence, if $m = -\frac{a}{b}$ and $d = \frac{c}{b}$, then

$$L = \{(x, mx + d) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

□

Definition 3.3.

(a) Suppose L is nontrivial and nonvertical, thus determined by an equation of the form $y = mx + d$ for some $m, d \in \mathbb{R}$ for every $(x, y) \in L$. m is called the **slope** of the line L. If $(x, y) \in L$, the slope of L determines how much y changes given a change in x. More specifically, for any real x, y, a, $(x, y) \in L$ if and only if $(x + a, y + ma) \in L$. The **vertical intercept** of L is the number d. The point $(0, d)$ lies on the line L, as well as the vertical axis.

(b) Two nonvertical lines are **parallel** if they have the same slope.

(c) Any two vertical lines are **parallel**.

(d) If m_1 is the slope of a nonvertical line L_1, m_2 is the slope of another nonvertical line L_2, we say L_1 and L_2 are **perpendicular** if $m_1 m_2 = -1$.

(e) Any vertical line is **perpendicular** to any horizontal line and vice versa.

(f) Three points are **colinear** if they all are points on the same line.

■

If one approaches this topic geometrically, one would usually start with the Pythagorean Theorem or one of its equivalent statements and be able to prove the slopes of nonvertical perpendicular lines are negative reciprocals of each other. Here, we take this as a definition and use it to prove the Pythagorean Theorem.

Exercise 3.4. Given a line L and a point not on the line C, show there exists a unique line M such that $C \in M$ and M and L are perpendicular. If L is nonvertical, consider the slope of L, if L is vertical, then M must be horizontal.

3.1 Linear systems

Definition 3.5. a **2 by 2 linear system of equations** is the union of two nontrivial lines. A **solution** of a linear system of equations is the intersection of the pair of lines forming the system. ■

Theorem 3.6. A solution of a 2 by 2 linear system is either a line, the empty set or a single point.

Proof. Suppose
$$L_1 = \{(x, y) \in \mathbb{R}^2 \mid a_1x + b_1y = c_1\}$$
and
$$L_2 = \{(x, y) \in \mathbb{R}^2 \mid a_2x + b_2y = c_2\}$$
for some real $a_1, b_1, c_1, a_2, b_2, c_2$. We seek a description of $L_1 \cap L_2$. If $(x, y) \in L_1 \cap L_2$, then (x, y) is a point for which
$$a_1x + b_1y = c_1$$
and
$$a_2x + b_2y = c_2,$$
then by properties of algebra and equality,
$$(a_1 - a_2)x + (b_1 - b_2)y = c_1 - c_2.$$
Informally, we've subtracted the second equation from the first. Hence, (x, y) lies on the line determined by $a_1 - a_2, b_1 - b_2$ and $c_1 - c_2$. We have a few cases.

- If $a_1 \neq a_2$, then
$$x = \frac{-(b_1 - b_2)y + c_1 - c_2}{a_1 - a_2}.$$
Since we also have $a_1x + b_1y = c_1$, it follows
$$a_1 \cdot \frac{-(b_1 - b_2)y + c_1 - c_2}{a_1 - a_2} + b_1y = c_1.$$
Hence,
$$\frac{-a_1(b_1 - b_2)y + b_1(a_1 - a_2)y}{a_1 - a_2} = (c_1 - a_1) \cdot \frac{c_1 - c_2}{a_1 - a_2}.$$
Hence,
$$\frac{-a_1b_1y + a_1b_2y + b_1a_1y - b_1a_2y}{a_1 - a_2} = (c_1 - a_1) \cdot \frac{c_1 - c_2}{a_1 - a_2}.$$
Hence,
$$\frac{(a_1b_2 - b_1a_2)y}{a_1 - a_2} = (c_1 - a_1) \cdot \frac{c_1 - c_2}{a_1 - a_2}.$$
Hence,
$$(a_1b_2 - b_1a_2)y = c_1(a_1 - a_2) - a_1(c_1 - c_2).$$
Hence,
$$(a_1b_2 - b_1a_2)y = a_1c_2 - c_1a_2.$$
We now have two subcases.

- If $a_1b_2 - b_1a_2 \neq 0$, then
$$y = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}.$$
And since $x = \frac{-(b_1-b_2)y+c_1-c_2}{a_1-a_2}$, it follows
$$x = \frac{-(b_1 - b_2) \cdot \frac{a_1c_2-c_1a_2}{a_1b_2-b_1a_2} + c_1 - c_2}{a_1 - a_2}.$$
Hence,
$$x = \frac{-(b_1 - b_2)(a_1c_2 - c_1a_2) + (c_1 - c_2)(a_1b_2 - b_1a_2)}{(a_1 - a_2)(a_1b_2 - b_1a_2)}.$$
Hence, if $a_1 \neq a_2$ and $a_1b_2 - b_1a_2 \neq 0$, the unique point which is an element of both L_1 and L_2 is
$$\left(\frac{-(b_1 - b_2)(a_1c_2 - c_1a_2) + (c_1 - c_2)(a_1b_2 - b_1a_2)}{(a_1 - a_2)(a_1b_2 - b_1a_2)}, \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}\right).$$
The condition $a_1b_2 - b_1a_2 \neq 0$ is equivalent to the statement the nontrivial lines L_1 and L_2 are not parallel.
- There are two subsubcases if $a_1b_2 - b_1a_2 = 0$. Recall
$$(a_1b_2 - b_1a_2)y = a_1c_2 - c_1a_2.$$
  * If $a_1c_2 - c_1a_2 \neq 0$, then we arrive at $0 \neq 0$, a contradiction. Hence, the existence of $(x, y) \in L_1 \cap L_2$ is challenged. Hence, $L_1 \cap L_2 = \emptyset$. That is, there are no solutions to this system. The conditions $a_1b_2 - b_1a_2 = 0$ and $a_1c_2 - c_1a_2 \neq 0$ are equivalent to the statement the nontrivial lines L_1 and L_2 are parallel and distinct.
  * Suppose $a_1c_2 - c_1a_2 = 0$. Since $a_1 \neq a_2$, by the nontriviality of both L_1 and L_2, the conditions $a_1b_2 - b_1a_2 = 0$, $a_1c_2 - c_1a_2 = 0$ and the Zero Product Property, it follows a_1 and a_2 are both not zero. We claim $L_1 \cap L_2 = L_1$, which implies the solution to this system is a line. Since $L_1 \cap L_2$ is a subset of L_1, we need only show L_1 is a subset of L_2. If
$$(x, y) \in L_1,$$
then
$$a_1x + b_1y = c_1.$$
Hence, $a_1a_2x + b_1a_2y = c_1a_2$. Since $a_1b_2 - b_1a_2 = 0$ and $a_1c_2 - c_1a_2 = 0$, it follows $a_1a_2x + a_1b_2y = a_1c_2$. Since $a_1 \neq 0$, $a_2x + b_2y = c_2$. Hence, $(x, y) \in L_2$. Since $(x, y) \in L_1$ by assumption, $(x, y) \in L_1 \cap L_2$. Hence, since (x, y) is an arbitrary element of L_1, it follows L_1 is a subset of $L_1 \cap L_2$. Hence, $L_1 \cap L_2 = L_1$. So, the solution is a line. Similarly, since $a_2 \neq 0$, we can show $L_1 \cap L_2 = L_2$. This now implies $L_1 = L_2$ in this case. The conditions $a_1b_2 - b_1a_2 = 0$ and $a_1c_2 - c_1a_2 = 0$ are equivalent to the statement the nontrivial lines L_1 and L_2 are parallel and not distinct.

- If $a_1 = a_2$, then
$$(b_1 - b_2)y = c_1 - c_2.$$
Again, some subcases.
  - $b_1 - b_2 = 0$.
    * $c_1 - c_2 = 0$. This means $L_1 = L_2$, so, $L_1 \cap L_2 = L_1$ in particular. Hence, the solution is a line.
    * $c_1 - c_2 \neq 0$. This means $0 \neq 0$. A contradiction. Hence, $L_1 \cap L_2 = \emptyset$. Hence, these lines are parallel and distinct.
  - $b_1 - b_2 \neq 0$. Then
$$y = \frac{c_1 - c_2}{b_1 - b_2}.$$
Since $a_1x + b_1y = c_1$, it follows
$$a_1x + b_1 \cdot \frac{c_1 - c_2}{b_1 - b_2} = c_1.$$
Hence,
$$a_1x = \frac{b_1c_2 - c_1b_2}{b_1 - b_2}.$$
    * If $a_1 \neq 0$, it follows
$$x = \frac{b_1c_2 - c_1b_2}{a_1(b_1 - b_2)}.$$
Hence, the unique point lying on both L_1 and L_2 is
$$\left(\frac{b_1c_2 - c_1b_2}{a_1(b_1 - b_2)}, \frac{c_1 - c_2}{b_1 - b_2}\right).$$
    * $a_1 = 0$.
      · $b_1c_2 - c_1b_2 \neq 0$. Then $0 \neq 0$. A contradiction. Hence, $L_1 \cap L_2 = \emptyset$. These lines are parallel and distinct.
      · $b_1c_2 - c_1b_2 = 0$. Since $b_1 \neq b_2$, again the conditions $a_1 = a_2 = 0$, $b_1c_2 - c_1b_2 = 0$, the nontriviality of L_1 and L_2 along with the Zero Product Property imply b_1 and b_2 are both not zero. Hence, if
$$(x, y) \in L_1,$$
then
$$b_1y = c_1.$$
Hence,
$$b_1b_2y = c_1b_2.$$
Hence,
$$b_1b_2y = b_1c_2.$$
Hence, since $b_1 \neq 0$,
$$b_2y = c_2.$$

Hence,

$$(x, y) \in L_2.$$

Hence, L_1 is a subset of $L_1 \cap L_2$. Hence,

$$L_1 \cap L_2 = L_1.$$

In this case again, since $b_2 \neq 0$ as well, $L_1 = L_2$. These lines are parallel and not distinct.

□

In summary, two lines either intersect at a single point, are the same line, or are parallel but distinct.

Exercise 3.7. Prove the above theorem in another way as follows. Let L_1 and L_2 be two nontrivial lines in the plane.

(a) Using Theorem 3.2, suppose L_1 is nonvertical. Write a general equation for it in terms of its slope m_1 and vertical intercept d_1.

  (i) If L_2 is also nonvertical, write a general equation for it in terms of its slope m_2 and vertical intercept d_2. There are now three cases.

    (A) If $m_1 \neq m_2$, what is $L_1 \cap L_2$?

    (B) If $m_1 = m_2$ and $d_1 = d_2$, what is $L_1 \cap L_2$?

    (C) If $m_1 = m_2$ but $d_1 \neq d_2$, what is $L_1 \cap L_2$?

  (ii) If L_2 is vertical, write a general equation for it using Theorem 3.2. What is $L_1 \cap L_2$ in this case?

(b) If L_1 is vertical, write a general equation for it using Theorem 3.2.

  (i) If L_2 is also vertical, there are two cases. Find $L_1 \cap L_2$ in each case.

  (ii) If L_2 is not vertical, write a general equation for it using Theorem 3.2. Find $L_1 \cap L_2$ in this case.

Definition 3.8.

(a) The **horizontal intercept** of a line L, if it exists, is an element of the intersection $L \cap \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, which is the intersection of L with the horizontal axis. If L is a function, a horizontal intercept of L is just a zero of L, if it exists.

(b) The **vertical intercept** of a line L, if it exists, is an element of the intersection $L \cap \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$, which is the intersection of L with the vertical axis.

■

Exercise 3.9.

If a is any real number,

1. What are the horizontal intercepts of the line $\{(a, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$?
2. What are the vertical intercepts of the line $\{(a, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$?
3. What are the horizontal intercepts of the line $\{(x, a) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$?
4. What are the vertical intercepts of the line $\{(x, a) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$?

Theorem 3.10. Two distinct points determine a unique nontrivial line.

Proof. If (x_1, y_1) and (x_2, y_2) are distinct, then either $x_1 \neq x_2$ or $y_1 \neq y_2$. We wish to find real a, b, c for which

$$ax_1 + by_1 = c$$

and

$$ax_2 + by_2 = c.$$

Hence, since the right-hand side of both of these equations is equal to c, we can reduce these two equations to a single equation by transitivity of equality. That is,

$$ax_1 + by_1 = ax_2 + by_2.$$

Hence,

$$-a(x_2 - x_1) = b(y_2 - y_1).$$

If $x_1 \neq x_2$, then $x_2 - x_1 \neq 0$. Hence, if $x_1 \neq x_2$, then

$$a = -b \cdot \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence,

$$ax_1 + by_1 = c$$

becomes

$$-bx_1 \cdot \frac{y_2 - y_1}{x_2 - x_1} + by_1 = c.$$

Hence,

$$b \cdot \left(-x_1 \cdot \frac{y_2 - y_1}{x_2 - x_1} + y_1 \right) = c$$

so that

$$b \cdot \left(\frac{-x_1(y_2 - y_1) + y_1(x_2 - x_1)}{x_2 - x_1} \right) = c.$$

Hence,

$$b \cdot \left(\frac{-x_1(y_2 - y_1) + y_1(x_2 - x_1)}{x_2 - x_1} \right) = c.$$

Hence,

$$b \cdot \left(\frac{-x_1 y_2 + x_1 y_1 + y_1 x_2 - y_1 x_1}{x_2 - x_1} \right) = c.$$

Hence,

$$b \cdot \left(\frac{-x_1 y_2 + y_1 x_2}{x_2 - x_1} \right) = c.$$

The equation

$$-b\cdot\frac{y_2-y_1}{x_2-x_1}x+by=b\cdot\frac{-x_1y_2+y_1x_2}{x_2-x_1}$$

determines a line which contain both (x_1,y_1) and (x_2,y_2). It seems we have a choice of b. If $b=0$, then this equation describes the entire plane, which is by definition a trivial line. Since we are seeking a nontrivial line, we choose $b\neq 0$. Hence, we can do some algebra and divide out by it. Also remember this is when $x_1\neq x_2$.

We claim the equation

$$-(y_2-y_1)x+(x_2-x_1)y=-x_1y_2+y_1x_2$$

determines a line containing both (x_1,y_1) and (x_2,y_2). And if (x_1,y_1) and (x_2,y_2) are distinct, then this equation determines the unique line which contains both of these points. Certainly,

$$(x_1,y_1),(x_2,y_2)\in\{(x,y)\in\mathbb{R}^2\mid -(y_2-y_1)x+(x_2-x_1)y=-x_1y_2+y_1x_2\}$$

by replacing x with x_1 and y with y_1 in the equation $-(y_2-y_1)x+(x_2-x_1)y=-x_1y_2+y_1x_2$. Same with (x_2,y_2). To show this equation determines the unique line containing both of these points if (x_1,y_1) and (x_2,y_2) are distinct, we again find real a,b,c for which

$$ax_1+by_1=c$$

and

$$ax_2+by_2=c,$$

in the same way as above. If $x_1\neq x_2$, we arrive at the same equation. If $x_1=x_2$, then $y_2\neq y_1$, the line is vertical and an equation is given by

$$x=x_1.$$

□

Corollary 3.11. If L is a nontrivial nonvertical line, m is its slope, d is its vertical intercept, $(x_1,y_1)\in L$ and $(x_2,y_2)\in L$, then

$$m=\frac{y_2-y_1}{x_2-x_1}$$

and

$$d=\frac{y_1x_2-x_1y_2}{x_2-x_1}.$$

Proof. The general equations are $y=mx+d$ and $-(y_2-y_1)x+(x_2-x_1)y=-x_1y_2+y_1x_2$ from Theorems 3.2 and 3.10. Two lines are the same if they are determined by the same numbers. □

Theorem 3.12. The Pythagorean Theorem If a,b,c,d,e,f are real numbers such that the three points (a,b), (c,d), (e,f) in the rectangular plane are not colinear, then the line determined by (a,b) and (c,d) and the line determined by (a,b) and (e,f) are perpendicular if and only if

$$(c-a)^2+(a-e)^2+(d-b)^2+(b-f)^2=(c-e)^2+(d-f)^2.$$

Proof. Suppose a, b, c, d, e, f are real numbers. Then

$$(c-e+d-f)^2 = (c-e)^2 + 2(c-e)(d-f) + (d-f)^2$$

Similarly,

$$\begin{aligned}(c-a+a-e+d-b+b-f)^2 =& (c-a+a-e)^2 + 2(c-e)(d-f) + (d-b+b-f)^2 \\ =& (c-a)^2 + 2(c-a)(a-e) + (a-e)^2 + 2(c-e)(d-f) \\ &+ (d-b)^2 + 2(d-b)(b-f) + (b-f)^2\end{aligned}$$

But $(c-e+d-f)^2 = (c-a+a-e+d-b+b-f)^2$. Hence,

$$\begin{aligned}(c-e)^2 + 2(c-e)(d-f) + (d-f)^2 =& (c-a)^2 + 2(c-a)(a-e) + (a-e)^2 + 2(c-e)(d-f) \\ &+ (d-b)^2 + 2(d-b)(b-f) + (b-f)^2\end{aligned}$$

Hence,

$$\begin{aligned}(c-e)^2 + (d-f)^2 =& (c-a)^2 + 2(c-a)(a-e) + (a-e)^2 \\ &+ (d-b)^2 + 2(d-b)(b-f) + (b-f)^2\end{aligned} \tag{3.13}$$

Now suppose the three points (a, b), (c, d), (e, f) in the rectangular plane are not colinear.

We have to consider two cases. First, the line determined by (a, b) and (c, d) and the line determined by (a, b) and (e, f) are both nonvertical. Hence, by Theorem 3.11, the slope of the line determined by (a, b) and (c, d) is

$$\frac{d-b}{c-a}$$

and the slope of the line determined by (a, b) and (e, f) is

$$\frac{b-f}{a-e}.$$

Then these two lines are are perpendicular if and only if

$$\frac{d-b}{c-a} \cdot \frac{b-f}{a-e} = -1$$

if and only if

$$(d-b)(b-f) = -(c-a)(a-e)$$

if and only if

$$(c-e)^2 + (d-f)^2 = (c-a)^2 + (a-e)^2 + (d-b)^2 + (b-f)^2$$

by Equation 3.13. Hence, if these two lines are nonvertical, the theorem if proved. Then if one of them is vertical, say the one determined by (a, b) and (c, d), then $a = c$. And this line is perpendicular to the line determined by (a, b) and (e, f) if and only if the line determined by (a, b) and (e, f) is horizontal. Hence, $b = f$. Hence, these two lines are perpendicular if and only if

$$(c-e)^2 + (d-f)^2 = (c-a)^2 + (a-e)^2 + (d-b)^2 + (b-f)^2$$

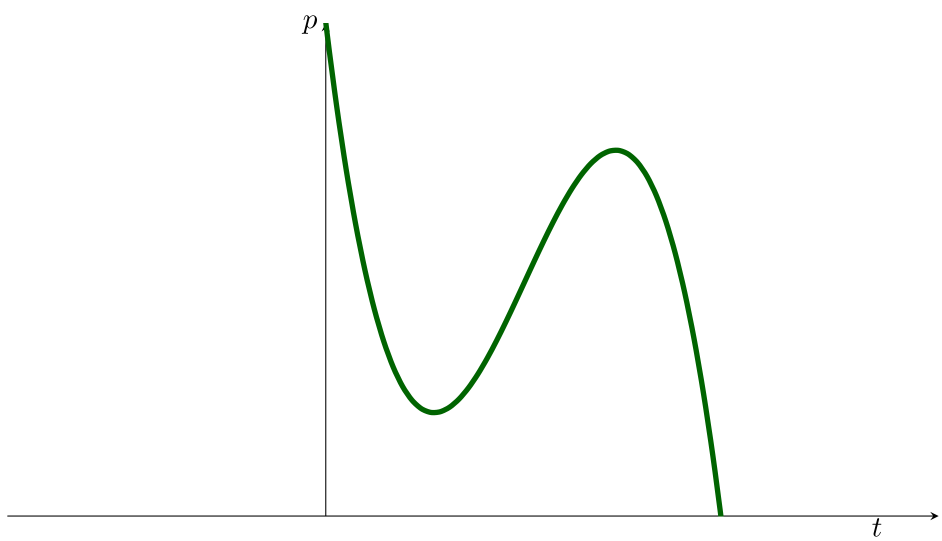
by Equation 3.13. In any case, the theorem is proved. □

4 Polynomials

Polynomials were first introduced in Definition 2.37. Affine functions, which are polynomials of degree less than 2, were discussed extensively in Theorems 2.44 and 2.53, Exercises 2.38, 2.45 and 2.54, and Section 3.

Below is drawing of the cubic (degree 3) polynomial

$$p(t) = -0.5(t-3.3)^3 - (t-3.3)^2 + 1.5(t-3.3) + 4.4.$$



4.1 Quadratic functions

Definition 4.1. The function $\mathrm{id}_{\mathbb{R}} \cdot \mathrm{id}_{\mathbb{R}}$ is special. Notice

$$\mathrm{id}_{\mathbb{R}} \cdot \mathrm{id}_{\mathbb{R}}(x) = \mathrm{id}_{\mathbb{R}}(x) \cdot \mathrm{id}_{\mathbb{R}}(x) = x \cdot x = x^2$$

for all x. We call $\mathrm{id}_{\mathbb{R}} \cdot \mathrm{id}_{\mathbb{R}}$ the **square function** and name it sq. Then

$$\mathrm{sq}(x) = x^2 \quad \text{if} \quad -\infty < x < \infty.$$

Notice $\mathrm{sq} = q_{1,0,0}$ as described in Definition 2.37. sq is the unique monomial of degree 2 with leading coefficient 1.

■

Theorem 4.2. (a) $\mathrm{sq}^{-1}((0,\infty)) = \mathbb{R} \setminus \{0\}$. That is, sq is positive on $(-\infty, 0) \cup (0, \infty)$.

(b) $\mathrm{sq}(0) = 0$. The only zero of sq occurs at 0.

(c) The range of sq is $[0, \infty)$.

(d) sq is strictly increasing on the closed interval $[0, \infty)$ and strictly decreasing on the closed interval $(-\infty, 0]$.

(e) sq has a local minimum value of 0 at 0. The minimum value of sq is 0.

(f) $\text{sq}(x) \to \infty$ as $x \to \infty$ and $\text{sq}(x) \to \infty$ as $x \to -\infty$. In particular, sq does not have a maximum value.

Proof. (a) $x \in \text{sq}^{-1}((0, \infty))$ if and only if $\text{sq}(x) > 0$ if and only if $x^2 > 0$ if and only if $x \neq 0$. This last step follows from Exercise 1.14. Since x is an arbitrary real number, $\text{sq}^{-1}((0, \infty)) = \mathbb{R} \setminus \{0\}$.

(b) $\text{sq}(0) = 0^2 = 0 \cdot 0 = 0$.

(c) Part (a) and (b) imply $\text{sq}^{-1}([0, \infty)) = \mathbb{R}$ and so, since $\mathbb{R}$ is the domain of sq, by Exercise 2.25 n, it follows $\text{range(sq)} = \text{sq}(\mathbb{R}) = \text{sq}(\text{sq}^{-1}([0, \infty)))$ is a subset of $[0, \infty)$. We now need to demonstrate $[0, \infty)$ is a subset of range(sq). This follows from Theorem B.16. If $y > 0$, then there is some positive x for which $x^2 = y$. This is the square root of y. Namely,

$$\text{sq}(\sqrt{y}) = (\sqrt{y})^2 = y.$$

Hence, every positive real number is in the range of sq. And since $\text{sq}(0) = 0$, it follows $[0, \infty)$ is a subset of range(sq). Since these sets are subsets of each other, they must be the same.

(d) If $0 \leq x_1 < x_2$, then $x_1 \geq 0$ and $x_1 < x_2$. Since multiplication on the left by a nonegative number preserves order,

$$\text{sq}(x_1) = x_1^2 = x_1 x_1 \leq x_1 x_2.$$

Similarly, multiplication on the right by a positive number preserves order, which implies

$$x_1 x_2 < x_2 x_2 = x_2^2 = \text{sq}(x_2).$$

Hence, by trichotomy,

$$\text{sq}(x_1) < \text{sq}(x_2).$$

That is, if $x_1 < x_2$ and are in the interval $[0, \infty)$, then $\text{sq}(x_1) < \text{sq}(x_2)$. Hence, sq is strictly increasing on the closed interval $[0, \infty)$. If $x_1 < x_2 \leq 0$, then $x_2 \leq 0$ and $x_1 < x_2$. Since multiplication on the left by a negative number is a decreasing function,

$$\text{sq}(x_1) = x_1^2 = x_1 x_1 > x_1 x_2.$$

Similarly, multiplication on the right by a nonpositive number is decreasing, which implies

$$x_1 x_2 \geq x_2 x_2 = x_2^2 = \text{sq}(x_2).$$

Hence, by trichotomy,

$$\text{sq}(x_1) > \text{sq}(x_2).$$

That is, if $x_1 < x_2$ and are in the interval $(-\infty, 0]$, then $\text{sq}(x_1) > \text{sq}(x_2)$. Hence, sq is strictly decreasing on the closed interval $(-\infty, 0]$.

(e) From parts (a) and (b), $\text{sq}(x) \geq 0 = \text{sq}(0)$ for all real x.

(f) Given $M > 0$, if $N = \sqrt{M}$, the square root of M, then $x^2 \geq M$ whenever $|x| \geq N$. This demonstrates $\text{sq}(x) \to \infty$ as $x \to \infty$ and $\text{sq}(x) \to \infty$ as $x \to -\infty$.

□

Recall the following important fact:

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd.$$

We conclude this by using the distributive property three times. Some people call this the foil method. But we'll refer to it as repeated distribution.

A special case occurs when computing squares of sums:

$$(a+b)^2 = a^2 + 2ab + b^2.$$

We now explain the following important identity for quadratic functions.

Theorem 4.3. Completing the Square

If a, b, c are real numbers and a is not zero,

- then, for any real number x,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

- then the function $q_{a,b,c}$ either
  - is strictly increasing on $\left[-\frac{b}{2a}, \infty\right)$, strictly decreasing on $\left(-\infty, -\frac{b}{2a}\right]$, and has a minimum value of $c - \frac{b^2}{4a}$ at the input $-\frac{b}{2a}$ if $a > 0$ or
  - is strictly decreasing on $\left[-\frac{b}{2a}, \infty\right)$, strictly increasing on $\left(-\infty, -\frac{b}{2a}\right]$, and has a maximum value of $c - \frac{b^2}{4a}$ at the input $-\frac{b}{2a}$ if $a < 0$.

Proof.

- It's easy to verify this if we start with the right hand side. The key is to use repeated distribution.

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}.$$

  Then distribute a.

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$= ax^2 + bx + \frac{b^2}{4a} + c - \frac{b^2}{4a}$$

$$= ax^2 + bx + c.$$

-
  - Case when $a > 0$:
    From the first item (completing the square),

$$q_{a,b,c} = l_{a,c-\frac{b^2}{4a}} \circ \mathrm{sq} \circ l_{1,\frac{b}{2a}}.$$

For this proof, let $f = l_{a,c-\frac{b^2}{4a}}$ and $g = l_{1,\frac{b}{2a}}$ for simplicity of notation. Then $q_{a,b,c} = f \circ \mathrm{sq} \circ g$.

Now, if $a > 0$, then f is increasing on $\mathbb{R}$ by Theorem 2.44. Also, sq is increasing on $[0, \infty)$ by Theorem 4.2. And g is increasing on $\mathbb{R}$ by Theorem 2.44 and since $1 > 0$. Now, by Theorem 2.46, $\mathrm{sq} \circ g$ is increasing on $g^{-1}([0,\infty)) \cap \mathbb{R} = g^{-1}([0,\infty)) = [-\frac{b}{2a}, \infty)$. $g^{-1}([0,\infty)) \cap \mathbb{R} = g^{-1}([0,\infty))$ since $g^{-1}([0,\infty))$ is a subset of $\mathbb{R}$. And $g^{-1}([0,\infty)) = [-\frac{b}{2a}, \infty)$ by Theorem 2.44.

Similarly, by Theorem 2.46, $f \circ (\mathrm{sq} \circ g)$ is increasing on $(\mathrm{sq} \circ g)^{-1}(\mathbb{R}) \cap [-\frac{b}{2a}, \infty) = [-\frac{b}{2a}, \infty)$. $(\mathrm{sq} \circ g)^{-1}(\mathbb{R}) \cap [-\frac{b}{2a}, \infty) = [-\frac{b}{2a}, \infty)$ since $(\mathrm{sq} \circ g)^{-1}(\mathbb{R}) = \mathbb{R}$ and $\mathbb{R} \cap [-\frac{b}{2a}, \infty) = [-\frac{b}{2a}, \infty)$. Hence, $q_{a,b,c}$ is increasing on $[-\frac{b}{2a}, \infty)$.

When considering where sq is decreasing, we obtain $q_{a,b,c}$ is decreasing on $\left(-\infty, -\frac{b}{2a}\right]$.

By Theorem 2.51, since $\mathrm{sq}(0) = 0$ is a local minimum value of sq at 0, and since $g(-b/2a) = 0$, 0 is a minimum value of $\mathrm{sq} \circ g$ at $-b/2a$. By Theorem 2.50, since f is increasing on $\mathbb{R}$, $f(\mathrm{sq}(g(-\frac{b}{2a}))) = c - \frac{b^2}{4a}$ is a minimum value of $q_{a,b,c} = f \circ (\mathrm{sq} \circ g)$ at $-\frac{b}{2a}$.

– The case when $a < 0$ is similar.

□

Exercise 4.4. If a, b, c are real numbers and $a < 0$, complete the square as in the above theorem to demonstrate $q_{a,b,c}$.

(a) is strictly decreasing on $\left[-\frac{b}{2a}, \infty\right)$,

(b) strictly increasing on $\left(-\infty, -\frac{b}{2a}\right]$ and

(c) has a maximum value of $c - \frac{b^2}{4a}$ at the input $-\frac{b}{2a}$.

Exercise 4.5. Prove the above theorem in the following way. If $a > 0$, and b, c and x are real numbers, then $ax^2 + bx + c = p + k$, where $p \geq 0$ and k is a real number.

(a) Find p in terms of a, b, c and x.

(b) Find k in terms of a, b, c.

(c) Conclude $ax^2 + bx + c \geq k$ for all $x \in \mathbb{R}$.

Exercise 4.6. Explain why the following statement is either true or false. There exists a least number b with the following property. The quadratic function $q_{1,b,4}$ has no zeros.

4.1.1 Square Roots and the zeros of quadratic functions

Recall Theorem B.16 and Definition 1.28. If $a \geq 0$, there's a unique nonnegative number b for which $b^2 = a$. We denote $b = \sqrt{a}$.

Exercise 4.7. Explain the given claim or answer the given question.

(a) $\sqrt{0} = 0$.

(b) If $a = n^2$, with n an integer, we say a is a **perfect square**. What is $\sqrt{a}$ in terms of n?

(c) What are the first 10 perfect squares?

(d) If a is not a perfect square, then $\sqrt{a}$ is irrational.

Theorem 4.8. The quadratic formula

Suppose a, b, c are real numbers and a is not zero.

(a) If $b^2 - 4ac > 0$, then $q_{a,b,c}$ has exactly two real zeros,

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

That is,

$$q_{a,b,c}^{-1}(\{0\}) = \left\{ \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\}.$$

In other words, a solution of the equation $ax^2 + bx + c = 0$ is either

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

When referring to either one of these solutions, we may write

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where the $\pm$ is read 'plus or minus'.

(b) If $b^2 - 4ac = 0$, then $q_{a,b,c}$ has exactly one zero,

$$\frac{-b}{2a}.$$

That is, if $b^2 = 4ac$, then

$$q_{a,b,c}^{-1}(\{0\}) = \left\{ \frac{-b}{2a} \right\}.$$

(c) If $b^2 - 4ac < 0$, then $q_{a,b,c}$ has no real zeros.

Proof. These follow from completing the square in Theorem 4.3. For all x,

$$q_{a,b,c}(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

The reader can fill in the details. For their endeavours, we remark $\sqrt{a^2} = |a|$ for every real a. □

Exercise 4.9. Prove Theorem 4.8.

Theorem 4.10. If a quadratic polynomial q has real zeros, say x_1 and x_2, which might be the same, suppose $x_1 \le x_2$ and q's leading coefficient is a, then

1. the intervals on which q is positive is
   (i) $(-\infty, x_1) \cup (x_2, \infty)$ if $a > 0$
   (ii) (x_1, x_2) if $a < 0$.
2. the intervals on which q is negative is
   (i) $(-\infty, x_1) \cup (x_2, \infty)$ if $a < 0$
   (ii) (x_1, x_2) if $a > 0$.

If $x_1 = x_2$, then the open interval (x_1, x_2) is empty. It contains no real numbers. And if $x_1 = x_2$, then $(-\infty, x_1) \cup (x_2, \infty)$ is a punctured line. The real line with one point removed. If q has no real zeros, then q is either positive or negative on $\mathbb{R}$, depending on a.

Proof. Again this follows directly from completing the square. Here's another method of proof. By the Factor Theorem (Theorem 4.18),

$$q(x) = a(x - x_1)(x - x_2)$$

for all real x.

Now suppose $a < 0$.

If $x < x_1$, then $x < x_2$ by Trichotomy, since $x_1 \le x_2$. Hence, $x - x_1 < 0$ and $x - x_2 < 0$. Remember the product of two negative numbers is positive. But remember a is negative. So, if $x < x_1$, then

$$q(x) = a(x - x_1)(x - x_2) < 0.$$

The same argument works when $x > x_2$. Now, if $x_1 < x < x_2$, then $x - x_1 > 0$, $x - x_2 < 0$, so that $(x - x_1)(x - x_2) < 0$. Therefore, $q(x) = a(x - x_1)(x - x_2) > 0$ since $a < 0$. Therefore, if $a < 0$, then $q(x) < 0$ exactly when either $x < x_1$ or $x > x_2$. And $q(x) > 0$ exactly when $x_1 < x < x_2$, if such x exist.

The proof when $a > 0$ is similar. □

Exercise 4.11.

(a) Is there any real number c for which $q_{1,-(c+1),c}$ does not have real zeros?

(b) Find $a \in \mathbb{R}$ for which $q_{1,0,a}$ has

- no zeros,
- one zero, and
- two zeros.

Exercise 4.12. If $a, b, c \in \mathbb{Z}$ (are integers) and $a \neq 0$ such that $q_{a,b,c}^{-1}(\{0\})$ is a subset of $\mathbb{Q}$ (the rationals), explain why we can find integers a_1, a_2, x_1, x_2 such that

$$q_{a,b,c} = l_{a_1,x_1} \cdot l_{a_2,x_2}.$$

4.2 Division of polynomials

Theorem 4.13. Division of Polynomials

If f and d are polynomials and d is not the zero polynomial, then there are *unique* polynomials q and r with the properties

$$f = d \cdot q + r$$

and the degree of r is strictly less than the degree of d. We call f the **dividend**, d is called the **divisor**, q the **quotient**, and r the **remainder**.

■

This theorem is almost verbatim the division statement for the integers in the Division Lemma A.18. Hence, a proof is not required. We simply replace the absolute value of d with the degree of d. In this sense, we say the collection of polynomials with real coefficients share a similar algebraic structure with the integers.

Exercise 4.14. Suppose f is any polynomial.

(a) If d is a nonzero constant polynomial, what are the unique polynomials q and r for which $f = q \cdot d + r$ and the degree of r is strictly less than the degree of d?

(b) If the degree of f is equal to the degree of d, what are the unique polynomials q and r for which $f = q \cdot d + r$ and the degree of r is strictly less than the degree of d?

(c) If the degree of f is strictly less than the degree of d, what are the unique polynomials q and r for which $f = q \cdot d + r$ and the degree of r is strictly less than the degree of d?

(d) Construct polynomials f, d, q, r for which $f = q \cdot d + r$ and the degree of r is two less than the degree of d and the degree of q is equal to the degree of d.

Theorem 4.15. The remainder theorem
If f is any polynomial and k is a real number, then there is a unique polynomial q such that

$$f = q \cdot l_{1,-k} + \mathrm{const}_{f(k)}.$$

Proof. This follows directly from division of polynomials. Because the degree of $l_{1,-k}$ is one, from Theorem 4.13, there is a unique polynomial q and a unique constant r such that

$$f = q \cdot l_{1,-k} + r.$$

Then

$$f(k) = q(k)(k - k) + r = r.$$

□

Exercise 4.16.

(a) Find q in the conclusion of the remainder theorem.

(b) Generalize the remainder theorem to work for $l_{m,d}$ if $m \neq 0$ instead of only $l_{1,-k}$. In other words, what is the remainder after dividing f by $l_{m,d}$ if $m \neq 0$?

Definition 4.17. Just as for integers in Defintion A.17, we say a polynomial d is a **factor** of a polynomial f if there is some polynomial q for which $f = d \cdot q$.

Theorem 4.18. The factor theorem

The real number k is a zero of the polynomial f exactly when $l_{1,-k}$ is a linear factor of f.

Proof. This follows directly from the remainder theorem. If k is a zero of f, then $f(k) = 0$ by definition, so the remainder theorem says

$$f = q \cdot l_{1,-k}$$

for some polynomial q. The other direction is even easier. If $f = q \cdot l_{1,-k}$ for some polynomial q, then certainly $f(k) = 0$. That is, k is a zero of f. ☐

4.3 Some analytic theorems

Theorem 4.19. The Intermediate Value Theorem for Polynomials

Given a polynomial p and real numbers, x_1, x_2, if $p(x_1) < 0$ and $p(x_2) > 0$, then there is some c in between x_1 and x_2 such that $p(c) = 0$. That is, as a polynomial changes between positive and negative, it has a zero.

Theorem 4.20. The Extreme Value Theorem for Polynomials

If $a, b \in \mathbb{R}$, $a \leq b$, and p is a polynomial, then there is some $y \in [a, b]$ for which $p(y) \geq p(x)$ for all $x \in [a, b]$. There is also some $w \in [a, b]$ such that $p(w) \leq p(x)$ for all $x \in [a, b]$. That is, for any polynomial p, for any real numbers a, b, the restricted function $p|_{[a,b]}$ has both a maximum and minimum value.

For proofs of these theorems, refer to Theorems B.11, B.14 and their lemmas.

Exercise 4.21.

1. Give an example of a real function which is positive on the closed interval $[-1, 0]$ and is negative on the half-open interval $(0, 1]$.

2. Is it possible for this function to be a polynomial? Why or why not?

4.4 Local and global data

Lemma 4.22. The zeros of any polynomial are discrete

If p is a nonconstant polynomial and c is a zero of p, then there are some real numbers a, b such that $a < c < b$ and p has no zeros in $(a, b) \setminus \{c\}$.

Proof. Suppose p is a nonconstant polynomial and c is a zero of p. By the Factor Theorem, p has no more zeros than its degree, which is a natural number. Suppose all of the zeros of p are ordered in a list $c_1 < c_2 < \cdots < c_k$. Then define d to be the least number out of the set $\{c_{i+1} - c_i \mid i \in \{1, \ldots, n-1\}\}$. This number exists because this set is finite. Then define $b = c + d$ and $a = c - d$. If $r \in (a, b) \setminus \{c\}$ and $p(r) = 0$, then $r = c_i$ for some $i = 1, \ldots, n$. But then either $0 < c - c_i < d$ or $0 < c_i - c < d$. This is a contradiction. Hence, $p(r) \neq 0$ for all $r \in (a, b) \setminus \{c\}$. □

Definition 4.23. If p is a polynomial, c is a zero of p, and $p = q \cdot (\mathrm{id}_{\mathbb{R}}^n \circ l_{1,-c})$ for some polynomial q and natural number n such that $q(c) \neq 0$, then n is called the **multiplicity** of c in p. By the Factor Theorem, each zero of p has a unique multiplicity. Recall $\mathrm{id}_{\mathbb{R}}^n$ is called the monic monomial of degree n. Notice $\mathrm{id}_{\mathbb{R}}^n \circ l_{1,-c}(x) = (x - c)^n$ for all $x \in \mathbb{R}$.

■

Lemma 4.24. The data in between zeros

If p is a polynomial and c_1, c_2 are real numbers such that $c_1 < c_2$ and there are no zeros of p in the interval (c_1, c_2), then either $p(x) > 0$ for all $x \in (c_1, c_2)$ or $p(x) < 0$ for all $x \in (c_1, c_2)$.

Proof. This follows directly from the Intermediate Value Theorem B.11. If p changes between positive and negative in (c_1, c_2), then p must have a zero there by the IVT. By assumption, it doesn't have any zeros in this interval. So, p is either positive or negative on (c_1, c_2). □

Theorem 4.25. The local data near zeros depend on multiplicities

- If a zero c of a polynomial p has even multiplicity, and $c_1 < c < c_2$ are the surrounding zeros of p, (if one doesn't exist, then replace it with either $\pm\infty$), then either $p(x) > 0$ or $p(x) < 0$ for all x in the interval (c_1, c_2) except at $x = c$. This is to say, near a zero with even multiplicity, a polynomial is either positive or negative but not both. Think of sq, the square function.
- Near a zero with odd multiplicity, a polynomial changes from positive to negative or vice versa. Think of $\mathrm{id}_{\mathbb{R}}$, the identity on $\mathbb{R}$.

Proof. Suppose p is a polynomial, c is a zero of p with multiplicity $2k$ for some natural number k. Then $p(x) = q(x)(x - c)^{2k}$ for all $x \in \mathbb{R}$, for some polynomial q for which $q(c) \neq 0$. If $c_1 < c < c_2$ are zeros of p (or c_1 or c_2 is equal to $\pm\infty$) and there are no other zero in the open interval (c_1, c_2), then q has no zeros in the open interval (c_1, c_2). Hence, by the above lemma, q is either positive or negative in (c_1, c_2). Suppose q is positive in (c_1, c_2). Then p is positive on $(c_1, c_2) \setminus \{c\}$. Since $(x - c)^{2k} = ((x - c)^k)^2 > 0$ for all x, and the product of two positive numbers is positive, it follows $p(x) = q(x)(x - c)^{2k} > 0$ if $x \in (c_1, c_2) \setminus \{c\}$.

If q is negative on (c_1, c_2), then p is negative on $(c_1, c_2) \setminus \{c\}$, since the product of a negative number and a positive number is negative. □

Exercise 4.26. Prove part (b) in the above theorem.

Exercise 4.27.

1. Sketch the graph of a polynomial which has exactly two local maximum values, one local minimum value and three zeros. *Hint:* Think about what the multiplicity of at least one of the zeros has to be.

2. Find an equation for the polynomial found in part (a).

4.4.1 Global data of polynomials

If $x > 1$, then x^2 is quite large. x^2 is x times larger than x. There's a way of sorting a collection of n objects using about n^2 comparisons of the objects with $<$. This is the brute force method of sorting objects because we simply compare each object to every other one to guarantee it's correctly sorted. For example, sorting one hundred (10^2) objects, say alphabetically sorting files or books, would take about ten thousand (10^4) comparisons. With one thousand (10^3) objects it'd take about one million (10^6) comparisons. A home computer is estimated to perform about 10^8 comparisons per second. Therefore, with a brute force algorithm, sorting a collection of 1 billion (10^9) objects would take about 317 years ($317 \cdot 31557600$ seconds). Collections of data these days are easily this big, so a more efficient method is needed.

10^4 is 100 times greater than 100. While one million is one thousand times greater than one thousand. Compare this with linear growth. $m100$ is m times greater than 100, while $m10^3$ is also m times 10^3. We say quadratic functions grow more than linear functions. In another way, we say the change of a quadratic function is greater than any linear function the further away from zero they are. A sorting method on n objects which only needs n comparisons would take about 10 seconds, a massive improvement. In practice, we have to settle for sorting methods in between a quadratic and linear number of comparisons.

In these notes, it's enough to discuss the fact that polynomials are unbounded in the sense that there is no maximum of their absolute value considering all inputs. We're also interested in the direction of growth, whether negative or positive. Polynomials have what we call **infinite limits at infinity**.

Recall Subsubsection 2.6.2.

Theorem 4.28. The leading coefficient and degree test

Given a polynomial p with leading coefficient a and degree n,

1. If $a > 0$ and n is even, then $\lim_{\infty} p = \infty$ and $\lim_{-\infty} p = \infty$.

2. If $a < 0$ and n is even, then $\lim_{\infty} p = -\infty$ and $\lim_{-\infty} p = -\infty$.

3. If $a > 0$ and n is odd, then $\lim_{\infty} p = \infty$ and $\lim_{-\infty} p = -\infty$.

4. If $a < 0$ and n is odd, then $\lim_{\infty} p = -\infty$ and $\lim_{-\infty} p = \infty$.

Proof. It's easy to verify this claim for monomials, which are completely determined by their leading coefficient and degree. The rest of the proof follows from the next section. If k is a natural number and r is a real number, then

$$\lim_{\pm\infty} \text{const}_r \cdot (\text{recip} \circ \text{id}_{\mathbb{R}}^k) = 0$$

by Theorem 5.13. If $p = \sum_{k=0}^{n} \text{const}_{a_k} \cdot \text{id}_{\mathbb{R}}^k$ for some natural number n and constants $a_0, \ldots, a_n$ with $a_n \neq 0$, then

$$p|_{\mathbb{R}\setminus\{0\}} = \text{const}_{a_n} \cdot \text{id}_{\mathbb{R}}^n \cdot \left(1 + \sum_{k=1}^{n} \text{const}_{\frac{a_{n-k}}{a_n}} \cdot (\text{recip} \circ \text{id}_{\mathbb{R}}^k)\right)$$

By Exercise 5.15 a and Theorem 5.13,

$$\lim_{\pm\infty} \left(1 + \sum_{k=1}^{n} \text{const}_{\frac{a_{n-k}}{a_n}} \cdot (\text{recip} \circ \text{id}_{\mathbb{R}}^k)\right) = 1.$$

Then Exercise 5.15 d with this theorem for monomials proves the theorem in general. □

From item 3 or 4 and the Intermediate Value Theorem B.11, it follows every polynomial of odd degree has at least one real zero. By the leading coefficient and degree test, if a polynomial has odd degree, it at some point attains a negative value and at some other input attains a positive value. Then, by the Intermediate Value Theorem, it must have a real zero.

Exercise 4.29.

(a) Suppose p is a polynomial and c is the least zero of p. That is, for all $p(c) = 0$ and, for all $x < c$, $p(x) \neq 0$. Prove p is either strictly increasing or strictly decreasing on $(-\infty, c]$. Hint: Use the Factor Theorem, Lemma 4.24, and Exercise 2.47.

(b) Prove a similar result if c is the greatest zero of p.

(c) Prove, if p is a polynomial of even degree, then there is some real number a such that $p + \text{const}_a$ has at least two real zeros.

(d) Conclude a polynomial of even degree has either a maximum or minimum on $\mathbb{R}$ but not both. Hint: Use parts (a), (b) and (c), Exercise 2.49, and the Extreme Value Theorem on a large enough interval. Finally, use the leading coefficient and degree theorem.

Exercise 4.30. For the following, $b > 0$.

1. What're the infinite limits at infinity (the global data) of $q_{-1,0,b}$?
2. If N is a natural number, what is the least positive number a with the property

$$(\text{abs} \circ q_{-1,0,b})((-\infty, -a] \cup [a, \infty))$$

is a subset of

$$[10^N, \infty)?$$

---

5 rational functions

Definition 5.1. We define a function, with both domain and codomain the punctured line

$$(-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\},$$

called recip, for which

$$\text{recip}(x) = \frac{1}{x} \quad \text{if} \quad x \neq 0.$$

recip(x) is the reciprocal or multiplicative inverse of x, as long as $x \neq 0$. recip is not a polynomial. It's the prototype of a **rational function**. It has peculiar non-polynomial behavior near the zero of the denominator, namely, 0. Compare Theorem 5.5. It also resembles a constant as the inputs increase without bound. Compare Exercise 5.2.

■

Exercise 5.2. Recall Definition 2.52.

$$\lim_{\pm\infty} \text{recip} = 0.$$

There are two claims.

Definition 5.3. A function R is a **rational function** exactly when there exists polynomials p and nonzero polynomial d for which

$$R = p \cdot (\text{recip} \circ\ d).$$

p is called the **numerator** and d is called the **denominator** of R. Then, for any input x, $R(x) = \frac{p(x)}{d(x)}$. Also, it's domain is $d^{-1}(\mathbb{R} \setminus \{0\})$. This is the set of all x for which $d(x) \neq 0$. That is, the domain of a rational function consists of every real number which isn't a zero of its denominator. The codomain of any rational function is $\mathbb{R}$.

■

5.1 Local Data

The local data of a rational function is comprised of the zeros of the rational function, their multiplicities, and the limits at the zeros of its denominator.

Definition 5.4. One-sided Infinite Limits at a point

Suppose f is a real function and a is a real number.

- We say

  $$\lim_{a^+} f = \infty$$

  to mean, for all real numbers M, there exists a positive real number δ (Greek letter delta) for which $f((a, a+\delta))$ is a nonempty subset of (M, ∞). This is said to be an **infinite right-side limit at a point**.

- We say

  $$\lim_{a^+} f = -\infty$$

  to mean, for all real numbers M, there exists a positive real number δ (Greek letter delta) for which $f((a, a+\delta))$ is a nonempty subset of $(-\infty, M)$. This is again an **infinite right-side limit at a point**.

- Similarly, we say

  $$\lim_{a^-} f = \infty$$

  to mean, for all real numbers M, there exists a positive real number δ (Greek letter delta) for which $f((a-\delta,a))$ is a nonempty subset of (M,∞). This is said to be an **infinite left-side limit at a point**.

- We say

  $$\lim_{a^-} f = -\infty$$

  to mean, for all real numbers M, there exists a positive real number δ (Greek letter delta) for which $f((a-\delta,a))$ is a nonempty subset of $(-\infty,M)$. This is said to be an **infinite left-side limit at a point**.

- If a function has a one-sided infinite limit at a point a, then we say the vertical line

  $$\{(x,y) \in \mathbb{R}^2 \mid x = a\}$$

  is a **vertical asymptote** of the rational function and say the rational function has a vertical asymptote at a.

Compare with Definition 2.52.

Theorem 5.5.

(a)
$$\lim_{0^+} \text{recip} = \infty.$$

We say recip approaches infinity from the right of zero. Or that infinity is the right-side limit of recip at zero.

(b)
$$\lim_{0^-} \text{recip} = -\infty.$$

We say recip approaches negative infinity from the left of zero. Or that negative infinity is the left-side limit of recip at zero.

Hence, the vertical axis is a vertical asymptote of recip.

Proof.

(a) We need to demonstrate
$$\lim_{0^+} \text{recip} = \infty.$$

Fix a positive real number M. Then define $\delta = \frac{1}{M}$. Then $\delta > 0$ and if $0 < x < \delta$, then $x < \frac{1}{M}$. Hence, $\frac{1}{x} > M$. Hence, $\text{recip}(x) > M$. If $M \leq 0$, then we can choose δ to be any positive real number, since $\text{recip}(x) > 0 \geq M$ for all positive x (Exercise 1.14). Since M is arbitrary, the claim is proved.

(b) This is similar. Fix a negative real number M and define $\delta = -\frac{1}{M}$. Then $\delta > 0$ and if $0 < 0 - x < \delta$, then $-\frac{1}{x} > -M$. Hence, $\text{recip}(x) < M$. Again, if $M \geq 0$, then we may choose any positive δ, since $\frac{1}{x} < 0 \leq M$ for every $x < 0$. Since M is arbitrary, the claim is proved.

□

Theorem 5.6.

(a) A rational function has a one-sided infinite limit at a if and only if the multiplicity of a as a zero of its denominator is greater than the multiplicity of a as a zero of its numerator.

(b) If R is a rational function and has a vertical asymptote at a, then the difference between the multiplicity of a as a zero of R's denominator and the multiplicity of a as a zero of R's numerator is even if and only if R approaches either infinity or negative infinity on both sides of a. This difference is odd if and only if R approaches infinity on one side of a and negative infinity on the other.

Proof. For this proof, suppose R is a rational function with numerator f and nonzero denominator d, and a is a real number.

(a) There are four cases,

$$\lim_{a^{\pm}} R = \pm\infty.$$

Suppose, without loss of generality,

$$\lim_{a^{+}} R = \infty.$$

The other three cases are similar. We first note $a \notin \text{domain}(R)$. That is, we must have $d(a) = 0$. Suppose

$$\lim_{a^{+}} R = \infty$$

but

$$d(a) \neq 0.$$

Then choose an interval $[b, c]$ containing a which does not contain any zeros of d. This is possible because the zeros of a polynomial are discrete. By the Extreme Value Theorem, since d is a polynomial and contains no zeros in $[b, c]$, there is some real $m > 0$ such that $|d(x)| \geq m$ if $b \leq x \leq c$. By the Extreme Value Theorem, since f is a polynomial, there is some $M > 0$ for which $|f(x)| \leq M$ if $b \leq x \leq c$. But since $\lim_{a^+} R = \infty$, there is some $\delta > 0$, which is less than $c - a$, such that $R(x) > \frac{M}{m}$ if $0 < x - a < \delta < c - a$. So, if $x \in (a, a + \delta)$, then $x \in [b, c]$. Hence, if $x \in (a, a + \delta)$, then $|R(x)| = R(x) > \frac{M}{m}$. But also if $x \in (a, a + \delta)$, then $|R(x)| = \frac{|f(x)|}{|d(x)|} \leq \frac{M}{m}$. This is a contradiction. Hence, if $R(x) \to \infty$ as $x \to a^+$, then $d(a) = 0$.

(b) Part (b) is analogous to Theorem 4.25.

□

Exercise 5.7. Construct an example of functions f and g and real number a for which f has an infinite limit at a but $f \cdot g$ does not have an infinite limit at a.

Theorem 5.8. The zeros of a rational function are the zeros of its numerator which aren't also zeros of its denominator.

Exercise 5.9. Prove Theorem 5.8.

Exercise 5.10.

(a) If a is a real number, does the rational function

$$l_{1,-a} \cdot (\text{recip} \circ q_{1,-2a,a^2})$$

have any infinite limits at a point?

(b) If a is a real number, does the rational function

$$q_{1,-2a,a^2} \cdot (\text{recip} \circ l_{1,-a})$$

have any infinite limits at a point?

(c) Does the rational function

$$l_{1,0} \cdot (\text{recip} \circ q_{1,0,1})$$

have any infinite limits at a point?

Theorem 5.11. Suppose b is a real number, n is a natural number, p and d are polynomials, b is not a zero of d, and R is a rational function for which $R(x) = \frac{(x-b)^n p(x)}{(x-b)^n d(x)}$ for all $x \neq b$ and $d(x) \neq 0$. If the domain of R is D, namely, all x for which $x \neq b$ and $d(x) \neq 0$, and if R^* is the rational function

$$R^*(x) = \begin{cases} R(x) & \text{if } d(x) \neq 0 \\ \frac{p(b)}{d(b)} & \text{if } x = b, \end{cases}$$

then

$$R^*|_D = R.$$

In this case, R is said to have a **removable singularity** at b, and R^* removes it. Notice the domain of R^* is $D \cup \{b\}$.

Exercise 5.12. Prove, if p is a polynomial and a is a real number, then the rational function

$$(p \circ l_{1,a} - \text{const}_{p(a)}) \cdot \text{recip}$$

has a removable singularity at 0. Recall, for every nonzero h, the quantity

$$(p \circ l_{1,a} - \text{const}_{p(a)}) \cdot \text{recip}(h) = \frac{p(a+h) - p(a)}{h}$$

is called the average rate of change of p between a and $a+h$. In the language of calculus, this exercise demonstrates any polynomial is differentiable at any real number. *Hint:* Use the Binomial Theorem A.16.

5.2 Global data

The division Theorem 4.13 allows us to split rational functions into two categories. In the first category, the degree of the numerator is strictly less than the degree of the denominator. This is true of the reciprocal. The second category consists of all other rational functions.

If R is a rational function with denominator d, then there are unique polynomials q and r with degree of r less than the degree of d such that

$$R = q + r \cdot (\text{recip} \circ\ d).$$

This is a restatement of the division Theorem 4.13. That is, we can always write a rational function as a polynomial plus a rational function in the first category. If the rational function is already in the first category, then of course $q = 0$.

Recall Subsubsection 2.6.2.

Theorem 5.13. If the degree of r is less than the degree of d, then

$$\lim_{\pm\infty} r \cdot (\text{recip} \circ\ d) = 0.$$

Corollary 5.14. If R is a rational function, then there is a unique polynomial q such that

$$\lim_{\pm\infty} (R - q) = 0.$$

Exercise 5.15. Suppose f and g are real functions.

(a) If

$$\lim_{\infty} f \in \mathbb{R}$$

and

$$\lim_{\infty} g \in \mathbb{R}$$

then

$$\lim_{\infty} (f + g) = \lim_{\infty} g + \lim_{\infty} f.$$

The same goes for limits at negative infinity. This is to say that finite limits at infinity are **additive**.

(b) If

$$\lim_{\infty} f \in \mathbb{R}$$

and

$$\lim_{\infty} g \in \mathbb{R}$$

then

$$\lim_{\infty} (f \cdot g) = \lim_{\infty} g \cdot \lim_{\infty} f.$$

The same goes for limits at negative infinity. This is to say that finite limits at infinity are **multiplicative**.

(c) If

$$\lim_{\infty} f \in \mathbb{R}$$

and

$$\lim_{\infty} f \neq 0,$$

then

$$\lim_{\infty}(\text{recip} \circ f) = \text{recip}(\lim_{\infty} f).$$

The same goes for limits at negative infinity. This is a consequence of the **continuity** of recip on $\mathbb{R} \setminus \{0\}$.

(d) If

$$\lim_{\infty} f \in \mathbb{R},$$

$$\lim_{\infty} f \neq 0$$

and

$$\lim_{\infty} g = \pm\infty$$

then

$$\lim_{\infty}(f \cdot g) = \pm\infty.$$

The same goes for limits at negative infinity.

(e) Construct examples of f and g for which $\lim_{\infty} f = 0$, $\lim_{\infty} g = \infty$ but $\lim_{\infty}(f \cdot g) = a$. Find similar f and g except $\lim_{\infty}(f \cdot g) = \infty$. This is why we call the value $\infty \cdot 0$ indeterminate.

(f) The following statements is an outline of Theorem 5.13. The details can be filled in if desired. Compare with the proof of Theorem 4.28. Suppose D is the domain of $r \cdot (\text{recip} \circ\, d)$. For all $x \in D$,

(I) If $r(x) = a_k x^k + \cdots + a_1 x + a_0$ and $d(x) = b_n x^n + \cdots + b_1 x + b_0$, with $k < n$, $a_k \neq 0$, and $b_n \neq 0$, then

$$r \cdot (\text{recip} \circ\, d)(x) = \frac{a_k x^k + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0}.$$

(II) Hence,

$$r \cdot (\text{recip} \circ\, d)(x) = x^{k-n} \cdot \frac{a_k + \cdots + \frac{a_1}{x^{k-1}} + \frac{a_0}{x^k}}{b_n + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}}$$

if $x \neq 0$.

(III) Hence,

$$r \cdot (\text{recip} \circ\, d)|_{D \cap \mathbb{R} \setminus \{0\}} = \text{id}^{k-n} \cdot \left(\text{const}_{a_k} + \sum_{j=1}^{k} \text{const}_{a_{k-j}} \cdot \text{id}^{-j} \right) \cdot \text{recip} \circ \left(\text{const}_{b_n} + \sum_{j=1}^{n} \text{const}_{b_{n-j}} \cdot \text{id}^{-j} \right)$$

(IV) Now,

$$\lim_{\infty} \mathrm{id}^{k-n} = 0$$

by Exercise 6.7 d.

(V) Similarly, by Exercises 2.54 b, 6.7 d and 5.15 a,

$$\lim_{\infty} \left(\mathrm{const}_{a_k} + \sum_{j=1}^{k} \mathrm{const}_{a_{k-j}} \cdot \mathrm{id}^{-j} \right) = a_k,$$

and

$$\lim_{\infty} \left(\mathrm{const}_{b_n} + \sum_{j=1}^{n} \mathrm{const}_{b_{n-j}} \cdot \mathrm{id}^{-j} \right) = b_n.$$

(VI) Hence, by Exercises 5.15 b and c, since $b_n \neq 0$,

$$\lim_{\infty} (r \cdot (\mathrm{recip} \circ d)) = 0 \cdot a_k \cdot \mathrm{recip}(b_n) = 0.$$

---

6 Invertibility and Radical Functions

Definition 6.1. We say a function is **one-to-one** on a set if no two different inputs in that set map to the same output. In symbols, f is one-to-one on A if, whenever $f(x) = f(y)$ and x and y are in A, then $x = y$. Equivalently, f is one-to-one on A if, for every $y \in f(A)$, there exists a unique $x \in A$ such that $f(x) = y$.

A mapping is **onto** if its codomain equals its function's range. If (C, f) is a mapping, then (C, f) is onto if and only if $C = \text{range}(f)$. Recall $\text{range}(f) = f(\text{domain}(f))$.

A mapping is **invertible** if it is onto and its function is one-to-one on its domain.

■

If we say a function is invertible, we mean it's one-to-one on its domain and its codomain is equal to its range. In practice, however, finding the range of a function is a nontrivial manner.

Theorem 6.2. If a function f is invertible, then there exists an invertible function, denoted f^{-1}, called the **inverse** of f, such that

(a) $\text{domain}(f^{-1}) = \text{range}(f)$.

(b) $\text{range}(f^{-1}) = \text{domain}(f)$.

(c) $f \circ f^{-1} = \text{id}_{\text{range}(f)}$

(d) $f^{-1} \circ f = \text{id}_{\text{domain}(f)}$.

Proof. Suppose f is an invertible function. We define the relation

$$f^{-1} = \{(y, x) \in \text{range}(f) \times \text{domain}(f) \mid (x, y) \in f\}$$

with domain $\text{range}(f)$ and codomain $\text{domain}(f)$. First, we demonstrate this relation f^{-1} is a function. We need to demonstrate, for every $y \in \text{range}(f)$, there exists a unique $x \in \text{domain}(f)$ such that $(y, x) \in f^{-1}$. Since f is one-to-one, for every $y \in \text{range}(f)$, there exists a unique $x \in \text{domain}(f)$ such that $f(x) = y$. That is, $(x, y) \in f$. Hence, $(y, x) \in f^{-1}$ by definition. So, f^{-1} is a function. f^{-1} is also invertible since, for every $x \in \text{domain}(f)$, there exists a unique $y \in \text{range}(f)$ such that $f(x) = y$. This is because f is a function. Hence, $f^{-1}(y) = x$. This means $\text{domain}(f) = \text{range}(f^{-1})$ and f^{-1} is one-to-one. Hence, f^{-1} is invertible. In fact, $(f^{-1})^{-1} = f$.

By definition, $f(x) = y$ if and only if $f^{-1}(y) = x$.

We now need only show (c) and (d).

If $y \in \text{range}(f)$, then there exists $x \in \text{domain}(f)$ such that $f(x) = y$. Hence, $f^{-1}(y) = x$. Hence,

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y = \text{id}_{\text{range}(f)}(y).$$

This is (c) by 2.12, since $y \in \text{range}(f)$ is arbitrary and their domains are the same.

If $x \in \text{domain}(f)$, then there exists $y \in \text{range}(f)$ such that $f(x) = y$. Hence, $f^{-1}(y) = x$. Hence,

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x = \text{id}_{\text{domain}(f)}(x).$$

This is (d) by 2.12, since $x \in \text{domain}(f)$ is arbitrary and their domains are the same. □

Exercise 6.3. Suppose f is a function.

(a) f is one-to-one on domain(f) if and only if, for every $y \in \text{codomain}(f)$, the preimage $f^{-1}(\{y\})$ contains at most one element.

(b) If f is one-to-one on domain(f), then there exists a function, linv$f$, called the **left-inverse** of f, with domain the codomain of f and codomain the domain of f, such that $\text{linv} f \circ f = \text{id}_{\text{domain}(f)}$.

(c) If codomain(f) = range(f), then there exists a function, rinv$f$, called the **right-inverse** of f, with domain the range of f, and codomain the domain of f, such that $f \circ \text{rinv} f = \text{id}_{\text{range}(f)}$.

(d) If f is invertible and A is a subset of its domain, then $(f|_A)^{-1} = f^{-1}|_{f(A)}$.

(e) If f and g are both invertible functions, demonstrate $f \circ g$ is invertible and find its inverse in terms of the inverses of f and g.

Theorem 6.4. test for one-to-oneness

If f is a real function and is either strictly increasing or strictly decreasing on an interval, then it's one-to-one on that interval.

Proof. If f is strictly increasing on an interval I and $f(x_1) = f(x_2)$ with x_1 and x_2 in I, then $x_1 = x_2$, since otherwise by Trichotomy we can compare them, say $x_1 < x_2$. Since f is strictly increasing, $f(x_1) < f(x_2)$. The same works if f is strictly decreasing. □

Exercise 6.5.

(a) The inverse of a strictly increasing function is also strictly increasing.

(b) Construct an example of a function which is neither increasing nor decreasing on its domain but is one-to-one there. Find its inverse.

(c) When is an affine function $l_{m,d}$, for real numbers m, d, invertible? What is its inverse in this case?

(d) Find all the affine functions which are their own inverse.

(e) For which real numbers a is $l_{1,a}$ invertible? For those a, find the inverse of $l_{1,a}$. $l_{1,a}$ is called addition on the right by a.

(f) For which real numbers a is $l_{a,0}$ invertible? For those a, find the inverse of $l_{a,0}$. $l_{a,0}$ is called multiplication on the left by a.

(g) If a, b, c are real numbers and $a > 0$, find the largest interval on which $q_{a,b,c}$ is strictly increasing by Theorem 4.3. Call this interval I. Then find the range of $q_{a,b,c}|_I$, the restriction of $q_{a,b,c}$ to I. Then find the inverse of $q_{a,b,c}|_I$ by using Exercise 6.3 e.

(h) Find the condition on real a, b, c, d for which the rational function $l_{a,b} \cdot (\text{recip} \circ l_{c,d})$ is invertible.

6.1 radical functions

Definition 6.6. By Theorem B.16 through Exercise B.22, in particular by Definition B.21, for every real number a, x^a is a positive real number for every positive x. Thus, for every nonzero real number a, we may define the function id^a with domain and codomain the unbounded open interval $(0, \infty)$, the positive reals, such that

$$\mathrm{id}^a(x) = x^a$$

for every $x > 0$. In particular, if a is a natural number, then $\mathrm{id}^a = \mathrm{id}^a_{\mathbb{R}}|_{(0,\infty)}$, a restriction of the monic monomial introduced in Definition 2.37. id^a is called the **power function with exponent** a.

■

Exercise 6.7. Suppose a is a nonzero real number.

(a) If $a > 0$, then id^a is strictly increasing on $(0, \infty)$, hence is one-to-one on its domain. Hint: Refer to Exercise B.22. Which part?

(b) If $a > 0$, then $\lim\limits_{\infty} \mathrm{id}^a = \infty$.

(c) If $a < 0$, then id^a is strictly decreasing on $(0, \infty)$, hence is one-to-one on its domain. Hint: Refer to the above exercise.

(d) If $a < 0$, then $\lim\limits_{\infty} \mathrm{id}^a = 0$.

(e) If $a < 0$, then $\lim\limits_{0^+} \mathrm{id}^a = \infty$.

(f) $\mathrm{id}^a((0, \infty)) = (0, \infty)$. That is, the codomain of id^a equals its range.

(g) Find the inverse of id^a as another power function.

(h) If a is an even natural number, what domain can we extend the rule of id^a to and maintain its invertiblity?

(i) If a is an odd natural number, what domain can we extend the rule of id^a to and maintain its invertiblity? Hint: Refer to Theorem B.17.

(j) If a is a negative integer for which $-a$ is odd, what domain can we extend the rule of id^a to and maintain its invertiblity?

(k) Explain why $\mathrm{id}^{\frac{1}{2}} \circ \mathrm{sq} = \mathrm{abs} \setminus \{(0, 0)\}$ as relations by way of Theorem 2.12.

(l) Find the domain, range and inverse of the function

$$R = l_{1,1} \circ \mathrm{id}^3 \cdot (\mathrm{recip} \circ l_{1,-8} \circ \mathrm{id}^3).$$

We've encountered many functions whose algebraic description depends on a single real number. For example, we define, for every real number a, the functions ar_a, ml_a, and pow_a by

$$\mathrm{ar}_a(x) = x + a \quad \text{if } -\infty < x < \infty$$

$$\mathrm{ml}_a(x) = ax \quad \text{if } -\infty < x < \infty$$
$$\mathrm{pow}^a(x) = x^a \quad \text{if } 0 < x < \infty.$$

ar_a is addition on the right by a, ml_a is multiplication on the left by a, and pow^a is raising to the ath power. We now introduce another class of functions parameterized by a single positive, nonone number.

7 exponential functions

Definition 7.1. From Definition B.21, if $a > 0$, $a \neq 1$, and x is real, then a^x is a positive real number. Moreover, from the global data of the exponential and continuity of the exponential (Lemma B.28), if y is any positive real number, then there is some x for which $a^x = y$. From Exercise B.22, if $a > 1$, and $x_1 < x_2$, then $a^{x_1} < a^{x_2}$. Hence, if $a > 1$, we may define a strictly increasing function on $\mathbb{R}$ whose range is the positives, $\exp_a$, with domain $\mathbb{R}$, codomain $(0, \infty)$, for which

$$\exp_a(x) = a^x \quad \text{if } -\infty < x < \infty.$$

$\exp_a$ is called the **exponential function with base** a. Recall from Definition B.21, a^x is read the xth power of a. a^x is a number with base a and exponent x.

If $0 < a < 1$, we can similarly define $\exp_a$ with domain $\mathbb{R}$, codomain $(0, \infty)$, for which

$$\exp_a(x) = a^x \quad \text{if } -\infty < x < \infty.$$

Recall if $0 < a < 1$, $x \in \mathbb{R}$, then $a^x = (a^{-1})^{-x}$. This function is strictly decreasing on $\mathbb{R}$ and its range is $(0, \infty)$.

In any case, if $a > 0$, and $a \neq 1$, $\exp_a$ defined above is invertible. We denote its inverse by $\log_a$. It's domain is the positive real numbers and its codomain, which is also its range, is all of $\mathbb{R}$. $\log_a$ is called the **logarithmic function** with base a.

$\log_a$ is defined by the property

$$\log_a(y) = x$$

if and only if

$$a^x = y$$

for every positive y.

7.1 Properties of exp and log

Fix $a > 0$ and $b > 0$, both not 1. For all real numbers s and t, for all positive numbers A and B,

$a^{\log_a(A)} = A$	$\log_a(a^s) = s$
$a^s \cdot a^t = a^{s+t}$	$\log_a(AB) = \log_a(A) + \log_a(B)$
$(a^s)^t = a^{st}$	$\log_a(A^t) = t \cdot \log_a(A)$
$(ab)^s = a^s \cdot b^s$	$\log_{ab}(a) + \log_{ab}(b) = 1$
$a^s = b^{s \log_b(a)}$	$\log_a(A) = \dfrac{\log_b(A)}{\log_b(a)}$
$a^0 = 1$	$\log_a(1) = 0$
$a^1 = a$	$\log_a(a) = 1$

The first four rows are from Exercise B.22. The last two are from A.9. The fact $\log_a(A) = \dfrac{\log_b(A)}{\log_b(a)}$ is called the **change of base formula**. It follows from the other properties of logarithms. It means it doesn't matter what base you choose. They're all the same function up to a scale factor.

Then global data of $\exp_a$ implies global and local data of $\log_a$.

If $a > 1$, then

$\lim_{-\infty} \exp_a = 0$	$\lim_{0^+} \log_a = -\infty$
$\lim_{\infty} \exp_a = \infty$	$\lim_{\infty} \log_a = \infty$

If $0 < a < 1$, then

$\lim_{-\infty} \exp_a = \infty$	$\lim_{\infty} \log_a = -\infty$
$\lim_{\infty} \exp_a = 0$	$\lim_{0^+} \log_a = \infty$

Exercise 7.2. Prove all of these facts. Especially the properties of the logarithm from the rules of exponents and the global data of both $\exp_a$ and $\log_a$.

Just as polynomials are characterized in terms of their average rate of change, so are exponential and logarithmic functions.

$$\frac{\exp_a(s+t) - \exp_a(s)}{t} = a^s \cdot \frac{a^t - 1}{t}$$

and

$$\frac{\log_a(A+B) - \log_a(A)}{B} = \log_a\left(\left(1 + \frac{B}{A}\right)^{1/B}\right).$$

The average rate of change of an exponential function is a multiple of the function itself. Contrast this with the average rate of change of a polynomial. Also, it turns out $\log_a\left(\left(1 + \frac{B}{A}\right)^{1/B}\right)$ resembles $\frac{1}{A}$.

Exercise 7.3. Use the rules of exp and log to demonstrate these identities of their average rate of change.

Exercise 7.4. Suppose a, b are positive, nonone real numbers, and m, n are nonzero real numbers, and k, d are real numbers.

(a) Find the inverse of $l_{n,k} \circ \exp_a \circ\ l_{m,d}$ as a composition involving linear functions.

(b) Use properties of exponents to write $l_{n,k} \circ \exp_a \circ\ l_{m,d}$ as $l_{s,u} \circ \exp_b \circ\ l_{w,v}$ for some real numbers s, u, w and v and where $|s| = 1$.

(c) In the above exercise, find b for which $w = 1$.

(d) Find $b, s, u \in \mathbb{R}$ such that $l_{n,k} \circ \exp_a \circ\ l_{m,d} = l_{s,u} \circ \log_b$.

(e) If $f = l_{n,k} \circ \log_b \circ l_{m,d}$ find $u, v, w \in \mathbb{R}$ such that $f = \log_u \circ l_{v,w}$.

Exercise 7.5.

(a) Find all real numbers t such that

$$\left(\frac{1}{\log_2 t}\right)^2 + \log_t 8 = 4.$$

(b) Find all real numbers t such that

$$\log_t(2) - \log_4(t) = \log_2(3t).$$

Exercise 7.6. Suppose b is a real number greater than one. Define

$$f = \exp_b \circ (-\operatorname{recip})|_{(0,\infty)} \cup \operatorname{const}_{(-\infty,0],0}.$$

Recall $-\operatorname{recip}|_{(0,\infty)}(x) = -\frac{1}{x}$ for all $x > 0$, the restriction of the negative reciprocal to $(0, \infty)$. And $\operatorname{const}_{(-\infty,0],0}(x) = 0$ if $x \leq 0$, the constant zero function on $(-\infty, 0]$.

(a) What is the domain of f?

(b) What is the range of f?

(c) Is f invertible? If not, explain why.

(d) What is the largest interval, I, on which f is one-to-one? Find the inverse of $f|_I$.

(e) What is the global data of f? That is, does f have any limits at infinity?

(f) Sketch the graph of f.

Exercise 7.7.

(a) If $A > 0$, $P > 0$, $0 < r < 1$, $n \in \mathbb{N}$, find t for which

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}.$$

Write as a single logarithm. This formula is called the **compound interest formula**.

(b) If A, a, k and r are positive numbers,find t for which

$$A = \frac{k}{1 + a \cdot e^{-rt}}.$$

Write as a single logarithm. This type of function is called a **logistic function**.

(c) If A, a, b, c are real numbers, $c \neq 0$, find t for which

$$A = a \cdot e^{-\frac{(t-b)^2}{2c^2}}.$$

Write as a single logarithm. This type of function is called a **Gaussian function**.

Example 7.8. A substance of amount m_0 changes by a factor of p_0 every t_0 units of time.

(a) How much will be present at time t?

(b) How much does the amount change after one unit of time?

(c) How long will it take for it to change by a factor of p_1?

(a) We define a function f by $f(t) =$ the amount of substance after t units of time. Then

$$f(0) = m_0.$$

$$f(t_0) = m_0p_0.$$

$$f(2t_0) = f(t_0)p_0 = m_0p_0p_0 = m_0p_0^2.$$

And

$$f(3t_0) = f(2t_0)p_0 = m_0p_0^2p_0 = m_0p_0^3.$$

In general, if n is a positive integer,

$$f(nt_0) = m_0p_0^n.$$

If $t = nt_0$, then $n = \frac{t}{t_0}$, so,

$$f(t) = m_0p_0^{\frac{t}{t_0}}.$$

(b) And

$$f(1) = m_0p_0^{\frac{1}{t_0}}.$$

(c) Using

$$f(t) = m_0p_0^{\frac{t}{t_0}},$$

we wish to find t for which

$$f(t) = m_0p_1.$$

Hence,

$$m_0p_0^{\frac{t}{t_0}} = m_0p_1.$$

So, that

$$p_0^{\frac{t}{t_0}} = p_1.$$

So that

$$\frac{t}{t_0} = \log_{p_0}(p_1).$$

So that

$$t = t_0\log_{p_0}(p_1).$$

■

Example 7.9. If the half-life of a substance is t_0 years, how much will have decayed after 1 year? The half-life of a substance is how long it takes for half of the substance to decay. Again let's use $f(t)$ to denote the amount of substance left after t years. Then

$$f(t_0) = \frac{f(0)}{2}.$$

$$f(2t_0) = \frac{f(t_0)}{2} = \frac{f(0)}{4}.$$

And

$$f(nt_0) = \frac{f(0)}{2^n}$$

if n is a positive integer. Again, extrapolating, if $t = nt_0$, then $n = t/t_0$ and

$$f(t) = f(0)\left(\frac{1}{2}\right)^{\frac{t}{t_0}}.$$

So, the substance will decay by a factor of $\left(\frac{1}{2}\right)^{\frac{1}{t_0}}$ after one year. ■

8 Symmetry and some conic sections

Definition 8.1. We now mention what it means for a function to be even or odd. A real function f is **even** if, whenever x is in the domain of f, so is $-x$, and $f(-x) = f(x)$. f is **odd** if instead $f(-x) = -f(x)$.

■

Exercise 8.2.

(a) What is the only function, defined on $\mathbb{R}$, which is both even and odd?

(b) If n is an even integer, $\mathrm{id}_{\mathbb{R}}^n$ is even.

(c) If n is an odd integer, $\mathrm{id}_{\mathbb{R}}^n$ is odd.

(d) If f and g are both even, then $f + g$ is even and $f \cdot g$ is even.

(e) If f and g are both odd, then $f + g$ is odd and $f \cdot g$ is even.

(f) If f is even and g is odd and neither of them are the zero function, then $f + g$ is neither even nor odd if it's not the empty function and $f \cdot g$ is odd.

(g) If f is any function and g is even, then $f \circ g$ is even.

(h) If f is even and g is odd, then $f \circ g$ is even.

(i) If f and g are odd, then $f \circ g$ is odd.

(j) If f is either even or odd, and c is a zero of f, then $-c$ is also a zero of f.

(k) If f is odd and $0 \in \mathrm{domain}(f)$, then $f(0) = 0$.

(l) If f is a function and $a \in \mathbb{R}$ such that $f \circ l_{1,a}$ is an odd function, then what is $f(a)$?

(m) If f is a real function with domain $\mathbb{R}$, there exist unique functions e and o such that e is even, o is odd, and $f = e + o$.

(n) If f is odd, then, for every $a \in \mathbb{R}$, $\mathrm{const}_a \cdot (f \circ l_{a,0}) = \mathrm{const}_{|a|} \cdot (f \circ l_{|a|,0})$.

Evenness and oddness are kinds of symmetry a real function may or may not possess. In general, we mean symmetry with respect to distance.

Definition 8.3.

(a) We define the **distance** between two points in the rectangular plane (x_1, y_1) and (x_2, y_2) by

$$D((x_1, y_1), (x_2, y_2)) := \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}.$$

D is a function with domain $\mathbb{R}^2 \times \mathbb{R}^2$ and codomain $\mathbb{R}$.

(b) We define the **distance** between a point (u, v) and a line L in the plane to be the minimum of the function $D|_{\{(u,v)\} \times L}$, which is the restriction of the distance function D to the subset of the plane $\{(u, v)\} \times L$.

■

Exercise 8.4. If $P = (u, v)$ is a point in the plane and $L = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$, determine that the minimum of the function $D|_{\{P\}\times L}$ exists. Hint: first consider the square of the distance function, $D \cdot D$, complete the square, and use Theorem 2.50 with $\mathrm{id}^{\frac{1}{2}}$ and $D \cdot D$.

Definition 8.5.

(a) Given a point P and a line L in the plane, let C be the point on L which is the minimum of $D|_{\{P\}\times L}$. We define the **reflection of** P **through** L to be the point $\mathrm{refl}(P, L)$ such that $\mathrm{refl}(P, L) \neq P$, $\mathrm{refl}(P, L)$ lies on the unique line which contains C and is perpendicular to L, and $D(P, C) = D(C, \mathrm{refl}(P, L))$.

(b) Given a point P and a point Q in the plane, let L be the unique line containing them both. We define the **reflection of** P **through** Q to be the point $\mathrm{refl}(P, Q)$ such that $\mathrm{refl}(P, Q) \neq P$, $\mathrm{refl}(P, Q) \in L$, and $D(P, Q) = D(Q, \mathrm{refl}(P, Q))$.

■

Exercise 8.6.

(a) Given a point P and a line L, show $\mathrm{refl}(P, L)$ is unique.

(b) Given a point P and a point Q, show $\mathrm{refl}(P, Q)$ is unique.

Definition 8.7.

(a) Given a subset S of the plane $\mathbb{R}^2$ and a line L in the plane, we say S is **symmetric about** L if, whenever P is a member of S, so is $\mathrm{refl}(P, L)$.

(b) Given a subset S of the plane $\mathbb{R}^2$ and a point Q in the plane, we say S is **symmetric about** Q if, whenever P is a member of S, so is $\mathrm{refl}(P, Q)$.

■

Exercise 8.8.

(a) Demonstrate a real function f is even if and only if it is symmetric about the vertical axis.

(b) Demonstrate a real function f is odd if and only if it is symmetric the origin O.

(c) If a, b, c are real numbers and $a \neq 0$, demonstrate $q_{a,b,c}$ is symmetric about the line $\{(x, y) \in \mathbb{R}^2 \mid x = -\frac{b}{2a}\}$.

(d) Define $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. S^1 is called the **unit circle**. Demonstrate S^1 is symmetric about the origin and any line which contains the origin.

(e) If a, b are nonzero real numbers, define $E(a, b) := \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, an ellipse centered at the origin. Demonstrate $E(a, b)$ is symmetric about the origin, the vertical axis, and the horizontal axis.

(f) If a, b are nonzero real numbers, define $H(a,b) := \{(x,y) \in \mathbb{R}^2 \mid -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, a hyperbola centered at the origin, opening in the vertical direction. Demonstrate $H(a,b)$ is symmetric about the origin, the vertical axis, and the horizontal axis.

(g) If a, b are nonzero real numbers, define $NC(a,b) := \{(x,y) \in \mathbb{R}^2 \mid \frac{x}{|a|} = \frac{y}{|b|}\}$, a null cone centered at the origin. Demonstrate $NC(a,b)$ is symmetric about the origin, the vertical axis, and the horizontal axis.

8.1 Hyperbolic Trigonometry

One sheet of the hyperbola $x^2 - y^2 = 1$ is traced out by the equations

$$x(t) = \cosh(t)$$

$$y(t) = \sinh(t)$$

if t is real. For all real t, $\cosh(t) = \frac{e^t + e^{-t}}{2}$ and $\sinh(t) = \frac{e^t - e^{-t}}{2}$. From here you can verify arithmetically that $\cosh^2(t) - \sinh^2(t) = 1$ for all real t. If you wish, cosh and sinh are the hyperbolic analogues of their elliptic cousins cos and sin. Just as cos and sin form parametric equations of a circle, cosh and sinh form **parametric equations** of a sheet of a hyperbola. The other sheet of $x^2 - y^2 = 1$ is traced out by $(-\cosh(t), \sinh(t))$ if $t \in \mathbb{R}$. Similarly, $(\sinh, \cosh)$ traces out the top sheet of $y^2 - x^2 = 1$.

Just as a circular angle can be thought of as a parameter for a circle, the argument t for sinh and cosh may be thought of as a hyperbolic angle.

Other hyperbolas are given by the equation $xy = a$ for any nonzero real a. These are rotations of $x^2 - y^2 = a^2$.

https://www.desmos.com/calculator/yskzxon6kb

8.2 Maps on the plane

Recall real-valued functions defined on an interval are those which we're used to. These are what we call real functions. Polynomials, the absolute value function, logarithms, powers, are all examples of this type of function. Here we introduce plane-valued maps *defined on the plane*. If T is such a function, we write $T : \mathbb{R}^2 \to \mathbb{R}^2$ to mean T is a function with domain $\mathbb{R}^2$ and whose codomain is $\mathbb{R}^2$.

After we fix an origin, we write $O = (0,0)$. If $P = (a,b)$ as a point in the plane, in rectangular coordinates, define

$$\begin{pmatrix} a \\ b \end{pmatrix} := \mathbf{OP}.$$

A vector with the correspondence between a point and a plane along with the origin is called a **column vector**.

The collection of vectors in the plane written as column vectors, along with these operations, is denoted by $\mathbb{R}^2$.

We add column vectors as

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}.$$

Scalar multiplication ends up being

$$k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}.$$

The norm or length of a column vector is

$$\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| = \sqrt{a^2+b^2}.$$

For example, the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y \end{pmatrix}$ describes a horizontal shift of the plane to the right by one. If f is a real-valued function defined on the real line, consider the parameterized curve c given by $c(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ if x in the domain of f. The range of c is the graph of f (as points in the plane). Then $T \circ c$ is a parameterized curve whose domain is the domain of f for which

$$T \circ c(x) = \begin{pmatrix} x+1 \\ f(x) \end{pmatrix}.$$

The range of $T \circ c$ is the graph of the function g given by $g(x) = f(x-1)$ for all x for which $x-1$ is in the domain of f.

For example, the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$ describes a horizontal dilation of the plane by a factor of 2. If f is a real-valued function defined on the real line, consider the parameterized curve c given by $c(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ if x in the domain of f. The range of c is the graph of f (as points in the plane). Then $T \circ c$ is a parameterized curve whose domain is the domain of f for which

$$T \circ c(x) = \begin{pmatrix} 2x \\ f(x) \end{pmatrix}.$$

The range of $T \circ c$ is the graph of the horizontal function g given by $g(x) = f\left(\frac{x}{2}\right)$ for all x for which $\frac{x}{2}$ is in the domain of f.

Similarly, the map $\text{Refl}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\text{Refl}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ describes the reflection of the plane about the y-axis.

If f is a real-valued function defined on the real line, consider the parameterized curve c given by $c(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ if x in the domain of f. The range of c is the graph of f (as points in the plane). Then $\text{Refl}_2 \circ c$ is a parameterized curve whose domain is the domain of f for which

$$\text{Refl}_2 \circ c(x) = \begin{pmatrix} -x \\ f(x) \end{pmatrix}.$$

The range of $\text{Refl}_2 \circ c$ is the graph of the horizontal function g given by $g(x) = f(-x)$ for all x for which $-x$ is in the domain of f. In particular, if f is even, then the ranges of $\text{Refl}_2 \circ c$ and the range of c are the same. Similarly, Refl_1 is the reflection of the plane about the x-axis.

Similarly, the map $S_k : \mathbb{R}^2 \to \mathbb{R}^2$ given by $S_k \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ \frac{y}{k} \end{pmatrix}$ describes a **squeeze mapping**.

These maps are interesting because, since $xy = a$ implies, if $S_k \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, then $s_1 s_2 = a$. That is, points which lie on a given hyperbola get sent to the same hyperbola under S_k. In particular, hyperbolas of the form $xy = constant$ are preserved under any squeeze mapping. This is analogous to rotations, which preserve circles. So, squeeze mappings may be thought of as hyperbolic rotations.

Squeeze mappings preserve hyperbolic areas. That is, areas which are formed by two straight lines emanating from the origin and the bounded section of the hyperbola $xy = a$ which is formed by the intersection of these two lines with the hyperbola.

9 Sequences and Series

Definition 9.1. A **sequence** is a function whose domain is a subset of the integers. Sequences and counting are highly correlated, which make them seemingly simple objects. The nth **term** of a sequence is the output of the sequence whose input is n. A **recursive sequence** is a sequence whose nth term is defined in terms of its earlier terms.

Example 9.2. The nth factorial is denoted by $n!$, defined recursively as

$$n! = n \cdot (n-1)!$$

and

$$0! = 1.$$

Example 9.3. An **arithmetic sequence** is the restriction of a linear function to the natural numbers. Any arithmetic sequence s then can be defined by two real numbers a, b for which

$$s = l_{a,b}|_{\mathbb{N}}.$$

Exercise 9.4. Describe any arithmetic sequence recursively.

Example 9.5. A **geometric sequence** is the restriction of an exponential function to the natural numbers.

Exercise 9.6. Describe any geometric sequence recursively.

Definition 9.7. A **series** is a sequence of partial sums. Given a sequence a_n, a **partial sum** S_n is defined recursively as

$$S_1 = a_1$$

and

$$S_{n+1} = a_{n+1} + S_n.$$

Then

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

We introduce **summation notation**. We write

$$S_n = \sum_{i=1}^{n} a_i.$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

Example 9.8. For example, if $a_n = 1+n$, then $S_3 = \sum_{i=1}^{3} a_i = a_1+a_2+a_3 = 1+1+(1+2)+(1+3) = 2+3+4 = 9$.

$$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) = 45.$$

$$\sum_{i=1}^{6} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$

$$= \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.71805555556.$$

This is close to the number e.

Exercise 9.9. (a) Find $\sum_{i=1}^{12} 5$.

(b) If $\sum_{i=1}^{n} a_i = 12$ and $\sum_{i=1}^{n} b_i = -15$, what is $\sum_{i=1}^{n}(a_i + b_i)$?

The i in summation notation is called the **index**. It doesn't matter what index we use. It's like a variable. From context, we do not mean the imaginary unit i.

Example 9.10. If $a_n = n$, then

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n.$$

Also,

$$S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Then $2S_n = n(n+1)$. Hence, $S_n = \frac{n(n+1)}{2}$. This method is how Carl Friedrich Gauss summed the integers from 1 to 100 in grade school, or so the story goes. $\frac{n(n+1)}{2}$ is the nth triangle number.

Example 9.11. An arithmetic series is a sequence of partial sums from an arithmetic sequence. If $a_n = cn + d$, then

$$S_n = \sum_{i=1}^{n}(ci + d) = c\left(\sum_{i=1}^{n} i\right) + dn = \frac{cn(n+1)}{2} + dn$$

$$= \frac{cn(n+1) + 2dn}{2} = \frac{n}{2}\left(cn + d + c + d\right) = \frac{n}{2}(a_1 + a_n).$$

Example 9.12. A geometric series is a sequence of partial sums from a geometric sequence. If $a_n = c(d^n)$, then

$$S_n = \sum_{i=0}^{n} c(d^i) = c\left(\sum_{i=0}^{n} d^i\right) = c \cdot \frac{1 - d^n}{1 - d}$$

if $d \neq 1$. Notice here we start the index at $i = 0$. If $|d| < 1$, then $d^n \to 0$ as $n \to \infty$. We write the limit of $S_n = \sum_{i=1}^{n} a_i$ as $n \to \infty$ as

$$\sum_{i=0}^{\infty} a_n.$$

If $|d| < 1$, then

$$\sum_{i=0}^{n} c(d^i) = c \cdot \frac{1 - d^n}{1 - d} \to \sum_{i=0}^{\infty} c(d^i) = \frac{c}{1 - d}$$

as $n \to \infty$.

We turn to the main application of this section. We relate decimal notation to fraction notation. Any rational number with a repeating decimal expansion can be written as the limit of a geometric series.

Example 9.13. For example, if $c = 3$, $d = \frac{1}{10}$, then

$$\sum_{i=0}^{\infty} \frac{3}{10^i} = \frac{3}{1 - \frac{1}{10}} = \frac{10}{3}.$$

Now notice

$$\frac{10}{3} = 1 + \frac{1}{3} = 1.33333... = 1.\overline{3}.$$

But this is exactly what the left hand side says.

Example 9.14. For example, $\frac{1}{9} = 0.\overline{1}$. Indeed,

$$0.\overline{1} = \sum_{i=1}^{\infty} \frac{1}{10^i} = \sum_{i=0}^{\infty} \frac{1}{10^i} - 1 = \frac{1}{1 - \frac{1}{10}} - 1 = \frac{1}{9}.$$

Alternatively, $\sum_{i=1}^{\infty} \frac{1}{10^i} = \sum_{i=0}^{\infty} \frac{1}{10^{i+1}} = \sum_{i=0}^{\infty} \frac{10^{-1}}{10^i}$.

As a surprising fact, we note $0.\overline{9} = 1$. This follows from above, since

$$0.\overline{9} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i} = \frac{9}{9} = 1.$$

Example 9.15. We can approximate irrational numbers with recursive sequences of rational numbers. For example, if

$$a_1 = 1$$

and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

then

$$a_2 = \frac{3}{2},$$

$$a_3 = \frac{17}{12},$$

and

$$a_4 = \frac{577}{408} \approx 1.414215686 \approx \sqrt{2}.$$

In fact, a_n can be made as close to $\sqrt{2}$ as we wish given we make n sufficiently large. That is, $a_n \to \sqrt{2}$ as $n \to \infty$.

This is a sequence of rational numbers approaching the irrational number $\sqrt{2}$. If the limit of a_n exists, then the limit of a_{n+1} exists, and if we write $a_n \to a$ as $n \to \infty$, then $a_{n+1} \to a$ as $n \to \infty$. Then $a = a/2 + 1/a$. Then $a^2 = 2$. Hence, $a = \sqrt{2}$. Of course, we need to show the limit exists and that $a_n > 0$ for all n, so we don't take the negative square root.

Exercise 9.16.

(a) Express

$$7 \cdot \sum_{i=0}^{\infty} 10^{-2i+2}$$

as a rational number.

(b) Express the rational number

$$\frac{4}{7}$$

as

$$a \cdot \sum_{i=0}^{\infty} 10^{m(i+1)}$$

for some integers a and m.

(c) Express the rational number

$$\frac{1}{7}$$

as

$$a + b \cdot \sum_{i=0}^{\infty} 10^{mi+n}$$

for some integers b, m and n, and partial sum a whose terms are a product of a digit with an integer power of 10 (a is a nonrepeating decimal).

10 Systems of equations and inequalities

Recall Subsection 3.1, Linear Systems.

Example 10.1. Finding the y-intercept of a line $ax + by = c$ is equivalent to solving the following system of equations

$$ax + by = c$$

and

$$x = 0.$$

To solve such a system, we simply **substitute** the second equation into the first to reduce the problem to a single equation with one unknown.

$$ax + by = c$$

and

$$x = 0$$

is equivalent to

$$by = c.$$

This may have one solution, no solutions, or infinitely many, depending on whether the line $ax+by = c$ describes a non-vertical line, a vertical line distinct from the y-axis, or the y-axis itself, respectively.

In general, a system given by

$$ax + by = c$$

and

$$dx + ey = f$$

for unknowns x, y and real numbers a, b, c, d, e, f, is said to be a **linear system of two equations in two variables**. Of course, these two equations separately describe lines. So, finding all pairs (x, y) which satisfy both of these equations is equivalent to finding where these lines intersect, if at all. Remember, two lines either intersect at a point, are parallel but distinct, or are the same line. Respectively, a linear system of two equations in two variables has either a unique solution, no solutions, or infinitely many solutions.

Example 10.2. The x-intercepts of a function given by an equation $y = f(x)$ are the solutions of the system

$$y = f(x)$$

and

$$y = 0.$$

Again, these two equations with two unknowns are equivalent to $f(x) = 0$, a single equation with one unknown. Depending on f, this equation can have any number of solutions. The system

$$y = f(x)$$

and

$$y = 0$$

is only a linear one if f is a linear function.

Example 10.3. Given two points (a, b) and (c, d) there are unique y_0 and k for which $b = y_0 k^a$ and $d = y_0 k^c$. These two equations form a **system of equations** with two unknowns, y_0 and k. Let's find them in terms of a, b, c, d. We use **elimination** by solving for one of the unknowns in both equations and reducing the problem to a single equation with one unknown.

We solve for y_0 first.

$$y_0 = b/k^a$$

and

$$y_0 = d/k^c.$$

Thus,

$$b/d = k^{a-c}.$$

Hence,

$$k = \left(\frac{b}{d}\right)^{\frac{1}{a-c}}.$$

Therefore,

$$y_0 = b\left(\frac{b}{d}\right)^{-\frac{a}{a-c}}.$$

So, if

$$f(x) = y_0 k^x,$$

with

$$k = \left(\frac{b}{d}\right)^{\frac{1}{a-c}}$$

and

$$y_0 = b\left(\frac{b}{d}\right)^{-\frac{a}{a-c}},$$

then

$$f(a) = b$$

and

$$f(c) = d$$

and f is the unique exponential function passing through the two points (a, b) and (c, d) with horizontal asymptote the horizontal axis.

■

Example 10.4. We can use this method to find the square root of a complex number. We ask, given a complex number $a + bi$, what is $\sqrt{a + bi}$ as a complex number? We want to find some real numbers u and v for which

$$\sqrt{a + bi} = u + vi.$$

Then

$$(u + vi)^2 = a + bi.$$

Then

$$u^2 - v^2 + 2uvi = a + bi.$$

Complex numbers are the same exactly when their real and imaginary parts agree. Then we have two equations

$$u^2 - v^2 = a$$

and

$$2uv = b$$

with two unknowns, u and v.

Then

$$u^2 = a + v^2$$

and

$$4u^2vu^2 = b^2.$$

We **substitute** this first equation into the second to end up with

$$4(a + v^2)v^2 = b^2.$$

This is one equation with one unknown. We have

$$4v^4 + 4av^2 - b^2 = 0.$$

Hence,

$$4(v^2)^2 + 4av^2 - b^2 = 0,$$

quadratic in v^2. Hence,

$$v^2 = \frac{-4a \pm \sqrt{16a^2 + 16b^2}}{8}$$

by the quadratic formula. Then

$$v^2 = \frac{-a \pm \sqrt{a^2 + b^2}}{2}.$$

Since $v^2 > 0$ if $v \neq 0$, $\frac{-a-\sqrt{a^2+b^2}}{2}$ is not a solution. Hence,

$$v^2 = \frac{-a + \sqrt{a^2 + b^2}}{2},$$

Using $u^2 = a + v^2$, it follows

$$u^2 = \frac{a + \sqrt{a^2 + b^2}}{2}.$$

We then have

$$|v| = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$$

$$|u| = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.$$

We choose

$$u = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.$$

And

$$v = \begin{cases} \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}} & \text{if } b \geq 0 \\ -\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}} & \text{if } b < 0. \end{cases}$$

These choices are arbitrary and $u + vi$ with this choice is called the **principal square root** of the complex number $a + bi$. The principal square root is a generalization of the convention that the square root of a positive number is also positive. When you get to trig, ask your instructor about the principal square root of a complex number in polar coordinates.

■

A The naturals, the integers, and the rationals

We know of two real numbers, 0 and 1, from Section 1, which both serve special purposes regarding addition and multiplication, respectively. We now describe three subcollections of them.

A.1 The natural numbers

A set is loosely thought of as a collection of objects. A **subset** of a set is a set itself entirely contained in the set.

Example A.1.

(a) The real numbers form a set.

(b) The **empty set** is the only set which contains no members, denoted $\emptyset$. The empty set is a subset of every set, including itself. The empty set contains no other subsets. Therefore, it's the only set which is a subset of every set.

■

We describe subsets of the real numbers. The first being the natural numbers. Remember addition takes two real numbers as inputs, and outputs a real number.

Definition A.2. 1 is the least **natural number** in that $1 < n$ for any other natural number n. And any other natural number is one more than some other natural number. That is, if n is a natural number and $1 < n$, then there is some other natural number m such that $n = m + 1$. Of course, with our algebra skills, we can find $m = n - 1$. The set of all natural numbers is denoted by $\mathbb{N}$.

■

The natural numbers have subsets of their own, and almost all such subsets share an important property.

Definition A.3. The Well-Ordering Principle

Every nonempty subset of natural numbers has a least element. That is, if S is a subset of natural numbers, then there is some natural number m which is contained in S, for which $m \leq n$ for all n contained in S.

Theorem A.4. The Principle of Mathematical Induction

If S is subset of $\mathbb{N}$, 1 is in S, and, whenever n is in S, $n+1$ is in S, it follows $S = \mathbb{N}$.

Proof. Suppose S is such a subset of the naturals. And suppose $S \neq \mathbb{N}$. That is, suppose there is some natural number m that is not in S. Denote all natural numbers not in S by $\mathbb{N} \setminus S$. Since m is in $\mathbb{N} \setminus S$, this set is nonempty. By the well-ordering principle, there is a smallest element m_0 in $\mathbb{N} \setminus S$. Hence, since $m_0 - 1 < m_0$ (and this is true because $0 < 1 = m_0 - (m_0 - 1)$), $m_0 - 1$ is in S (because if $m_0 - 1$ were not in S, then m_0 would not be the least element in $\mathbb{N} \setminus S$). But then, by a property of S, since $m_0 - 1$ is in S, $m_0 = m_0 - 1 + 1$ is in S. Hence, m_0 is both not in S and in S. This is a contradiction. Hence, our original assumption must be false, that $S \neq \mathbb{N}$. So, $S = \mathbb{N}$. □

Theorem A.5. $\mathbb{N}$ is discrete

For every natural number n, there is no natural number p for which $n - 1 < p < n$.

Proof. Suppose this is not true. Suppose there is some n and some p for which $n-1 < p < n$. Let S be the set of all natural numbers p for which there is some n for which $n-1 < p < n$. Then S is nonempty. So, by the well-ordering principle, there must be a smallest element, p_0. Then there is some n_0 for which $n_0 - 1 < p_0 < n_0$. Then there is no natural number q for which $p_0 - 1 < q < p_0$. This means $n_0 - 1 \leq p_0 - 1$. Hence, $n_0 \leq p_0$. But $p_0 < n_0$. This is not possible by trichotomy. Hence, our assumption must be false. That is, there are no natural numbers n and p for which $n-1 < p < n$. □

Definition A.6. For any natural number n and any real numbers $a_1, a_2, \ldots, a_n$, we write

$$\sum_{i=1}^{n} a_i := a_1 + a_2 + \cdots + a_n.$$

This is called **summation notation**. Again, associativity makes this unambiguous. Notice we may also write

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n,$$

which suggests we maybe be able to use induction to prove things about sums.

We reserve the right to start the sum at any **index**. For example, $\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$. ■

Theorem A.7. Linearity

If n is a natural number, $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ and c are real numbers, then

(a)

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

and

(b)

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i.$$

Proof. (a) We use induction. Let S be the set of all natural numbers n for which, for any given real numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, we have

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

We first show 1 is in S. Of course, for any given real numbers, a_1, b_1, $\sum_{i=1}^{1} (a_i + b_i) = a_1 + b_1$, $\sum_{i=1}^{1} a_i = a_1$ and $\sum_{i=1}^{1} b_i = b_1$. So,

$$\sum_{i=1}^{1} (a_i + b_i) = a_1 + b_1 = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i.$$

Now suppose n is in S. We must show $n+1$ is in S. Given any real numbers $a_1, \ldots, a_{n+1}$, $b_1, \ldots, b_{n+1}$, we have

$$\sum_{i=1}^{n+1}(a_i + b_i) = \sum_{i=1}^{n}(a_i + b_i) + a_{n+1} + b_{n+1},$$

$$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} a_i + a_{n+1}$$

and

$$\sum_{i=1}^{n+1} b_i = \sum_{i=1}^{n} b_i + b_{n+1}.$$

Since n is in S, it follows

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n+1}(a_i + b_i) = \sum_{i=1}^{n}(a_i + b_i) + a_{n+1} + b_{n+1} &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + a_{n+1} + b_{n+1} \\ &= \sum_{i=1}^{n} a_i + a_{n+1} + \sum_{i=1}^{n} b_i + b_{n+1} \\ &= \sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n+1} b_i.\end{aligned}$$

In conclusion, we have shown, 1 is in S and if n is in S, so is $n+1$. Hence, by the Principle of Mathematical Induction, $S = \mathbb{N}$. That is, for all n, if $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are real numbers, then

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

(b) This is left to the reader. □

Exercise A.8.

(a) Show, if n is a natural number, $a_1, \ldots, a_n$, and c are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

by induction.

(b) If n is a natural number, $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are real numbers, then

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i.$$

(c) If n is a natural number, express $\sum_{i=1}^{n} 1$ as a natural number. Show this by induction.

Definition A.9.

$$\begin{aligned}
2 &:= 1 + 1 \\
3 &:= 2 + 1 \\
4 &:= 3 + 1 \\
5 &:= 4 + 1 \\
6 &:= 5 + 1 \\
7 &:= 6 + 1 \\
8 &:= 7 + 1 \\
9 &:= 8 + 1 \\
10 &:= 9 + 1
\end{aligned}$$

The := means we are introducing new notation for convenience on the side of :. That is, the symbol 2 is defined to be the natural number $1 + 1$ and so on. Further, we define, for any natural number, n, for any nonzero real number a,

$$\begin{aligned}
a^0 &:= 1 \\
a^n &:= a^{n-1} \cdot a.
\end{aligned}$$

In particular, if $n = 1$, then $a^n = a^0 \cdot a = 1 \cdot a = a$.

If n is a natural number, we define $0^n := 0$. Just as 0^{-1} is undefined, so is 0^0. If a^n is a real number, a^n is called a **power** of a, and n is called the **exponent** of a^n, whilst a is the **base** of a^n. a^n is also called the **nth power of** a.

The numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are called **decimal digits**. We introduce **decimal notation**. That is, given any natural number n and any collection of decimal digits $a_0, a_2 \ldots, a_n$, we define

$$a_n a_{n-1} \cdots a_2 a_0 := \sum_{i=0}^{n} a_i 10^i.$$

The symbol $a_n a_{n-1} \cdots a_2 a_0$ is unfortunately similar to the product $a_n a_{n-1} \cdots a_2 a_0$. It should be clear from context what we mean. Decimal notation is simply a convenient way to write numbers which are relatively close to zero. For example, $17 = 1 \cdot 10^1 + 7 \cdot 10^0 = 10 + 7$. $100 = 1 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 10^0 = 10^2$. And $3141 = 3 \cdot 10^3 + 1 \cdot 10^2 + 4 \cdot 10 + 1$.

■

Exercise A.10. If a, b are real numbers and n, m are natural numbers,

(a) Prove, if $a = b$, then $a^n = b^n$ by induction.

(b) Prove, if $m = n$, then $a^m = a^n$ by induction on n.

(c) Prove $(ab)^n = a^n b^n$ by induction.

(d) Prove $a^{m+n} = a^m \cdot a^n$ by induction on n.

(e) Prove $(a^m)^n = a^{mn}$ by induction on n.

(f) Prove, if $0 < a < b$, then $a^n < b^n$ by induction. Deduce, if $a > 0$, $b > 0$ and $a^n < b^n$, then $a < b$.

(g) Construct an example of a and b real, natural number n for which $a^n = b^n$ but $a \neq b$.

(h) Prove, if $1 < a$ and $m < n$, then $a^m < a^n$. Hint: first prove if $a > 1$, then $a^n > 1$ by induction. Deduce, if $a > 1$ and $a^m < a^n$, then $m < n$.

Lemma A.11. If j is a natural number, then

$$1 + \sum_{i=0}^{j} 9 \cdot 10^i = 10^{j+1}.$$

Proof. By induction on j, if $j = 1$, then

$$1 + \sum_{i=0}^{j} 9 \cdot 10^i = 1 + \sum_{i=0}^{1} 9 \cdot 10^i = 1 + 9 \cdot 10^0 + 9 \cdot 10 = 10 + 9 \cdot 10 = (1+9) \cdot 10 = 10 \cdot 10 = 10^2.$$

Where did we use the distributive property?

If this formula is true for j, then

$$1 + \sum_{i=0}^{j+1} 9 \cdot 10^i = 1 + \sum_{i=0}^{j} 9 \cdot 10^i + 9 \cdot 10^{j+1} = 10^{j+1} + 9 \cdot 10^{j+1} = (1+9) \cdot 10^{j+1} = 10 \cdot 10^{j+1} = 10^{j+2}.$$

Hence, this formula is true for $j + 1$. By induction, this formula is proved. □

Theorem A.12. Any natural number has a unique decimal expansion.

Proof. By induction, 1 is of course written as a decimal. If n has a decimal expansion, say $n = a_k \cdots a_0 = \sum_{i=0}^{k} a_i 10^i$ for some natural number k and some digits $a_0, \dots, a_k$, if $a_0 < 9$, then $a_0 + 1 < 10$ is still a decimal digit, so $n + 1 = (a_0 + 1) \cdot 10^0 + \sum_{i=1}^{k} a_i 10^i$. So, $n = a_k \cdots a_1(a_0 + 1)$ as a decimal. If $a_0 = 9$, find the largest index j for which $a_0 = \cdots = a_j = 9$. This index exists by the well-ordering principle. Then

$$1 + \sum_{i=0}^{j} a_i 10^i = 1 + \sum_{i=0}^{j} 9 \cdot 10^i = 10^{j+1}.$$

This last equality follows from the above Lemma. Also, $a_{j+1} < 9$ by our choice of j, so $a_{j+1} + 1$ is still a decimal digit. Then

$$n + 1 = \sum_{i=0}^{j} 0 \cdot 10^i + (a_{j+1} + 1) 10^{j+1} + \sum_{i=j+2}^{k} a_i \cdot 10^i.$$

This representation is unique. If

$$\sum_{i=0}^{n} a_i 10^i = \sum_{i=0}^{m} b_i 10^i$$

for some natural numbers n, m and decimal digits a_i, b_i, then $n = m$ and $a_i = b_i$ if $i = 1, \dots, n$. □

Definition A.13. For each natural number n and k, with $k \leq n$, the number $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is called the **binomial coefficient**. Here and elsewhere, for any natural number n,

$$0! := 1$$
$$n! := n \cdot (n-1)!.$$

$n!$ is the **factorial** of n. It's the successive product of n with all smaller natural numbers.

■

Exercise A.14. For any natural numbers n and k with $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{A.15}$$

This does not need induction. It can be shown directly.

Theorem A.16. The Binomial Theorem

For any real numbers a and b, for any natural number n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Here, we use the convention, when the exponent does not depend on the base itself, that $0^0 = 1$.

Proof. Base case: If $n = 0$, then

$$(a+b)^0 = 1.$$

And

$$\sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^k = \binom{0}{0} a^{0-0} b^0 = \frac{0!}{0!(0-0)!} = 1.$$

Inductive hypothesis: Suppose

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Inductive step:

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

Now we perform a operation called **shifting the index**. It's like a change of variable. If $l = k+1$, then $k = l-1$, and if $k = 0$, then $l = 1$ and if $k = n$, then $l = n+1$. Then

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{l=1}^{n+1} \binom{n}{l-1} a^{n+1-l} b^l.$$

It doesn't matter what we call the index. We can replace l with k again to conclude

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^{k}+\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\\ &=\sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^{k}+\sum_{k=1}^{n+1}\binom{n}{k-1}a^{n+1-k}b^{k}\\ &=a^{n+1}+\sum_{k=1}^{n}\left(\binom{n}{k}+\binom{n}{k-1}\right)a^{n+1-k}b^{k}+b^{n+1}.\end{aligned}$$

Now we can use (A.15) to conclude

$$\begin{aligned}(a+b)^{n+1}&=a^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}a^{n+1-k}b^{k}+b^{n+1}\\ &=\sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^{k}.\end{aligned}$$

The last step holds since $\binom{n+1}{0}=1=\binom{n+1}{n+1}$. □

A.2 The integers

Definition A.17. 0 is an **integer**. Any natural number is an integer. The additive inverse of any natural number is an integer. Thus, the integers are the natural numbers, together with zero and their negatives. The collection of integers is denoted by $\mathbb{Z}$. $\mathbb{Z}$ is of course a subset of the real numbers. We say an integer p is a **factor** or **divisor** of another integer m if there is some integer n such that $m = pn$. In this case, we say m is a **multiple** of p and that p **divides** m. A **prime** integer is an integer which doesn't have any factors other than itself, 1 or -1.

■

Lemma A.18. The division Lemma

If f and d are integers and d is not zero, there are exist unique integers q and r for which

1. $f = dq + r$.
2. $0 \leq r < |d|$.

Since it is possible $d < 0$, we need to use the absolute value of d in item 2. f is called the **dividend**, d is called the **divisor**, q the **quotient** and r the **remainder.**

Proof. Suppose f and d are integers and d is not zero.

If $f \geq d$, then $f - d \geq 0$. Let C be the set of all nonnegative integers m for which there is a q such that $m = f - dq \geq 0$. This set is nonempty since $f - d$ is in C. Hence, by the well-ordering principle (well-ordering also works for subsets of nonnegative integers and not just subsets of positive integers), it follows there is a least such element in C, say r. Then there is some integer q for which $f = dq + r$. We need to show $0 \leq r < |d|$. If $r \geq |d|$, then $r - |d| = f - (q + \mathrm{sgn}(d))d \geq 0$. Hence, $r - |d|$ is in C. But $r - |d| < r$ since $d \neq 0$. Hence, this contradicts the leastness of r in C.

So, it must be the case that $r < |d|$. And by construction, $r \geq 0$. Hence, there are some integers q and r such that $f = dq + r$ and $0 \leq r < |d|$.

Now we consider the case when $f < d$. If $f < d$, then $-f > -d$. Hence, by the above case, there is a unique p and s such that $-f = -dp + s$ and $0 \leq s < |d|$. Then $f = dp + (-s)$. If $s = 0$, then $-s = 0$ and so

$$f = dp + 0.$$

So, we're done. If $s \neq 0$, then $-|d| < -s < 0$. We define $r = |d| - s$ and $q = p - \text{sgn}(d)$. Then

$$0 < r < |d|$$

and

$$f = dp - s = dp - |d| + |d| - s = d(p - \text{sgn}(d)) + r = dq + r.$$

$\text{sgn}(d) = 1$ if $d > 0$ or $\text{sgn}(d) = -1$ if $d < 0$.

We now show uniqueness. That is, suppose q and r are the integers satisfying $f = dq + r$ and $0 \leq r < |d|$. If there are some integers q' and r' such that $f = dq' + r'$ and $0 \leq r' < |d|$, then $dq + r = dq' + r'$. Hence, $r - r' = d(q - q')$. Now, since $0 \leq r < |d|$ and $0 \leq r' < |d|$, it follows $-|d| < r - r' < |d|$. That is, $|r - r'| < |d|$ by Exercise 1.17. But since $q - q'$ is an integer, if $q - q' \neq 0$, then $|q - q'| \geq 1$. Hence, $|d(q - q')| \geq |d|$. But since $r - r' = d(q - q')$, it follows $|r - r'| = |d(q - q')|$. Hence, we can't have both $|r - r'| < |d|$ and $|d(q - q')| \geq |d|$ by Trichotomy. Therefore, $q' = q$. Therefore, $r = r'$. This shows uniqueness. □

Exercise A.19. Demonstrate, with an example, there do not exist unique integers q, r such that $f = 0 \cdot q + r$.

Definition A.20. The **greatest common divisor** of two integers a and b is the unique integer d such that d divides a and b and, if m is a positive integer which divides a and b, then $m \leq d$. By convention, the greatest common divisor of 0 and 0 is 0.

■

Exercise A.21. Show if one of the integers a or b is not zero, then their greatest common divisor is positive.

Theorem A.22. Bézout's identity

If a and b are integers with greatest common divisor d, then there exist integers x and y such that $ax + by = d$.

Proof. Let a and b be integers. If $a = b = 0$, then $0 = 0 \cdot 0 + 0 \cdot 0$. So, we're done. Otherwise, one of a or b is not zero. Without loss of generality, $a \neq 0$. We let C be the set of all positive integers m for which there are integers x and y for which $m = ax + by$. Then C is not empty since $a \neq 0$, which means $|a| > 0$ and $|a| = a\,\text{sgn}(a) + b \cdot 0$. Hence, $|a|$ is in C. Hence, by the well-ordering principle, there is a least element of C, say, d. We show d is the greatest common divisor of a and b. First, there are some x and y for which $d = ax + by$. Now, use the division lemma to write $a = dq + r$ for some integers q and r. Then $r = a - dq = a - (ax + by)q = a(1 - x)q + b(-yq)$. Hence, r is in C. But since d is the least positive integer in C, it follows $r = 0$. This means d

divides a. Similarly, d divides b. We need to show d is the *greatest* common divisor. If m is an integer which divides both a and b, then there are integers q and q' for which $a = mq$ and $b = mq'$. Hence, $d = ax + by = mxq + myq' = m(xq + yq')$. Therefore, m divides d. In particular, $m \leq d$. □

Lemma A.23. Euclid's Lemma For integers a, b, p, if p is prime and p divides ab, then either p divides a or p divides b.

Proof. Suppose a and b are integers. First suppose $p > 0$. Suppose p divides ab. If p divides a, then we're done. Otherwise, since p is prime, the greatest common divisor between p and a is 1. Then by Bézout's identity, there are integers x and y such that $1 = ax + py$. Hence, $b = abx + pby$. Since p divides ab, it follows from this equation that p divides b.

□

Theorem A.24. The Fundamental Theorem of Arithmetic

Every positive integer is either prime or can be written as a product of prime integers.

Proof. The positive integers is another name for the natural numbers. Suppose there is a positive integer which is not prime and cannot be written as the product of prime numbers. Then, by the well-ordering principle, there is a smallest such number, say n. Since n is not prime, there are some positive integers $1 < a, b$, such that $n = ab$. Hence, since $1 < b$, and $0 < a$, it follows $a < ab = n$. Similarly, $b < n$. Hence, since n is the smallest positive integer which can't be written as the product of primes, a and b can be written as the product of primes. So, their product, namely n, can be written as a product of primes. Hence, every positive integer can be written as a product of prime integers. □

Exercise A.25. Show there are an infinite number of prime numbers. Hint: consider the opposite, that the list of primes terminates. Say $p_1, \ldots, p_n$ are all of the prime numbers. Consider the product of them plus one: $p_1 \ldots p_n + 1$.

A.3 The rational numbers

Definition A.26. The rational numbers, $\mathbb{Q}$ consist of all integers together with their multiplicative inverses. That is, every rational number can be written as ab^{-1} with integers a and b with b not zero. a is called the **numerator** of ab^{-1}, and b is its **denominator**. Remember, using different notation, $\frac{a}{b} = ab^{-1}$. Notice that every natural number is an integer, and every integer is a rational number. Thus, every natural number is a rational number. That is, $\mathbb{N}$ is a subset of $\mathbb{Z}$ and $\mathbb{Z}$ is a subset of $\mathbb{Q}$. But not the other way around. There are rational numbers which aren't natural numbers or even integers. And there are integers which aren't natural numbers.

■

Theorem A.27. Every rational number can be written in lowest terms. That is, if q is a rational number, then there exist integers a and b for which a and b have no common factors (other than 1 or -1) and $q = ab^{-1}$.

Proof. By Exercise 1.11 a and d, we can write any rational number ab^{-1} with a an integer and with b a natural number (a positive integer). Let S be the collection of all denominators of rational numbers which can't be written in lowest terms. If S is nonempty, then there is a least such natural number b by the Well-Ordering Principle. Then there is some integer a for which a and b have at least one common factor which isn't -1 or 1, say p. p is nonzero because b is nonzero. We can assume p is positive by Exercise 1.11 a. Also, $1 < p$ since 1 is always trivally a common factor of any two numbers. Since p is a factor of both a and b, ap^{-1} and bp^{-1} are integers. bp^{-1} is a natural number since b and p are both positive. We have $ap^{-1}(bp^{-1})^{-1} = ap^{-1}pb^{-1} = ab^{-1}$ by Exercise 1.9. Again, ab^{-1} cannot be written in lowest terms. So, neither can $ap^{-1}(bp^{-1})^{-1}$ since they are equal. Therefore, bp^{-1} is in S. But $bp^{-1} < b$ since $1 < p$ and by Exercise 1.14. But b is assumed to be the least member of S. This is a contradiction. So, our assumption that S is nonempty must be false. That is, every rational number can be written in lowest terms. □

The rational numbers satisfy all properties of addition, multiplication, and the order relation $<$. So, how are they different from the real numbers? The real numbers consist of all rational numbers together with the **irrational numbers**. An irrational number is a real number which isn't rational. This might seem circular. And it is because we haven't defined the real numbers yet at all. We only said what properties we'd like them to have. If you're convinced that the natural numbers exist, you might be convinced that the integers and even the rational numbers exist. Irrational numbers, and hence the rest of the real numbers, are more difficult to describe. However, we can verify the following fact.

Theorem A.28. The square root of 2 is not rational. That is, there does not exist a rational number q such that $q^2 = 2$.

Proof. On the contrary, suppose there does exist a rational number q such that $q^2 = 2$. Now by definition, there exist integers a and b for which b is not zero and $q = ab^{-1}$. Moreover, by the above Theorem, a and b have no common factors other than 1 or -1. Then $2 = (ab^{-1})^2 = a^2(b^{-1})^2$ by Exercise A.10. Hence, by Properties of Algebra,

$$a^2 = 2b^2.$$

This shows 2 is a factor of a^2 (a^2 is **even**). So, it must also be a factor of a by Euclid's Lemma. Hence, $a = 2k$ for some integer k. Hence, $2^2k^2 = a^2 = 2b^2$. Hence, $b^2 = 2k^2$. Hence, 2 is a factor of b^2, which means it's also a factor of b. We've shown 2 is both a factor of a and b, which contradicts the assumption that they share no common factors other than 1 and -1. Hence, our original assumption, that the number q for which $q^2 = 2$ is rational, must be false. That is, the square root of 2 is not rational. □

Notice we are careful to say the square root of 2 is not rational instead of irrational. We have not introduced the defining property of the real numbers yet.

B Some continuity results

Definition B.1. A real number x is called an **upper bound** of a set S if $s \leq x$ for all s in S. A set is **bounded from above** if an upper bound of it exists.

■

Definition B.2. The **real numbers** is the collection of objects with addition, multiplication, and the order relation described in the first section, along with the following condition on its subsets. If S is a nonempty, bounded from above subset of the real numbers, then there exists a real number x which is the **least upper bound** of the set S. That is, x is an upper bound of S and for every upper bound y of S, $x \leq y$. The property that every nonempty, bounded from above subset of the real numbers has a least upper bound is called the **least upper bound property**.

The set of all real numbers is denoted by $\mathbb{R}$ or $]-\infty, \infty[$.

■

Exercise B.3.

(a) There is at most one upper bound for every subset of the real numbers.

(b) x is the least upper bound of a subset S if and only if x is an upper bound of S and for all $\varepsilon > 0$, there exists some s in S for which $x - \varepsilon \leq s$. ε is the Greek letter epsilon.

(c) Prove the Well-Ordering Principle, Theorem A.3, from the least upper bound property.

Exercise B.4. 1. If a and b are real, find the least upper bound for each of the eight sets in Definition 1.26. If one doesn't exist, explain why.

2. Describe the square root of 2 as the least upper bound of a subset of rational numbers.

Lemma B.5. The Archimedean property of $\mathbb{R}$

If a and b are real numbers and $0 < a < b$, then there exists a natural number n for which $b < na$.

Proof. Let
$$S = \{c \in \mathbb{R} \mid c = na \text{ for some } n \in \mathbb{N} \text{ and } c \leq b\}.$$

Then S is nonempty since $1 \cdot a = a$ and $a < b$ by assumption. Hence, $a \in S$. S is bounded from above since $b \geq c$ for all $c \in S$ by definition. Hence, b is an upper bound of S. Hence, by the least upper bound property of $\mathbb{R}$, a least upper bound of S exists, say m. Then by Exercise B.3, there is some $n \in \mathbb{N}$ such that $m - a \leq na$. Hence, $m \leq (n+1)a$. If $(n+1)a \leq b$, then $(n+1)a \in S$. Hence, $m = (n+1)a$, since $m \leq (n+1)a$ and m is the least upper bound of S. But then this implies $(n+2)a > b$, since $(n+2)a = (n+1)a + a = m + a > m$, and so $(n+2)a \notin S$. Now if $(n+2)a > b$, then the Archimedean property is proved. If $(n+1)a > b$, then the Archimedean property is proved. □

Theorem B.6. The density of $\mathbb{Q}$

If a and b are real numbers and $a < b$, then there exists a rational number q such that $a < q < b$.

Proof. If $a < b$, then $b - a > 0$. Either $b - a > 1$ or $b - a \leq 1$. In any case, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $1 < n(b-a)$. Then $\frac{1}{n} < b - a$.

Suppose now $a \geq 0$. The case when $a < 0$ will be dealt with later. If $a \geq 0$, then either $\frac{1}{n} < a$ or $a \leq \frac{1}{n}$. Again by the Archimedean property and the Well-Ordering Principle, there is a least $m \in \mathbb{N}$ such that $a < \frac{m}{n}$. By the leastness of m, $\frac{m-1}{n} \leq a$. We claim $\frac{m}{n} < b$. Otherwise, $b \leq \frac{m}{n}$. Then
$$b - a \leq b - \frac{m-1}{n} \leq \frac{1}{n}.$$

But recall $\frac{1}{n} < b - a$. This is a contradiction. Hence, $\frac{m}{n} < b$. Since $a < \frac{m}{n}$, if we let $q = \frac{m}{n}$, then $a < q < b$.

If $a < 0$ and $b < 0$, then $-a > -b > 0$. Then, by the above argument, there exists a rational $-q$ such that $-b < -q < -a$. Hence, $a < q < b$.

Now, if $a < 0$ and $b > 0$, then $a < 0 < b$. Since 0 is rational, we let $q = 0$. Then $a < q < b$. □

B.1 Continuity of Polynomials

Definition B.7. If c and r are real numbers, and $r > 0$, define **the open interval of radius r, center c**

$$I_r(c) :=]c - r, c + r[.$$

Define **the closed interval of radius r, center c**

$$\overline{I_r(c)} := [c - r, c + r].$$

■

Exercise B.8.

(a) If c, δ are real numbers and $\delta > 0$, then $x \in I_\delta(c)$ if and only if there exists some $r \in \mathbb{R}$ such that $x = c + r$ and $|r| < \delta$.

(b) If c and r are real numbers, then $x \in I_r(c)$ if and only if $|x - c| < r$.

Lemma B.9. The continuity of polynomials

If p is a polynomial and c is a real number, then, for all $\varepsilon \in \mathbb{R}$, if $\varepsilon > 0$, then there exists some $\delta \in \mathbb{R}$ such that $\delta > 0$ and the image $p(I_\delta(c))$ is a subset of the open interval $I_\varepsilon(p(c))$.

Proof. Suppose p is a polynomial, c is a real number, and ε is a positive real number. Since p is a polynomial, there exists a natural number n and real numbers $a_1, \ldots, a_n$ such that $a_n \neq 0$ and $p(x) = \sum_{i=0}^{n} a_i x^i$ for all $x \in \mathbb{R}$. By the Binomial Theorem, for every $r \in \mathbb{R}$,

$$\begin{aligned} p(c+r) &= \sum_{i=0}^{n} a_i (c+r)^i \\ &= \sum_{i=0}^{n} a_i \sum_{k=0}^{i} \binom{i}{k} c^{i-k} r^k \\ &= \sum_{i=0}^{n} a_i \sum_{k=0}^{i} \binom{i}{k} c^{i-k} r^k \\ &= p(c) + \sum_{i=1}^{n} a_i \sum_{k=1}^{i} \binom{i}{k} c^{i-k} r^k. \end{aligned} \tag{B.10}$$

Now, similar to the proof of Lemma B.15, we define δ to be either $\frac{1}{2}$ or $\frac{\varepsilon}{\sum_{i=1}^{n}|a_i|\sum_{k=1}^{i}\binom{i}{k}|c|^{i-k}}$, whichever is less than or equal to the other. The number $\sum_{i=1}^{n}|a_i|\sum_{k=1}^{i}\binom{i}{k}|c|^{i-k}$ is positive because $a_n \neq 0$ and all other terms are nonnegative. Hence, since $\varepsilon > 0$ as well, $\delta > 0$. By the above exercise, for any $x \in \mathbb{R}$, $x \in I_\delta(c)$ if and only if $x = c + r$ for some $r \in \mathbb{R}$ such that $|r| < \delta$. If $|r| < \delta$, then $|r|^k < \delta^k$ for all natural numbers k by Exercise A.10 g. And by Exercise A.10 h, since $0 < \delta < 1$, $\delta^k < \delta$ for all natural numbers $k > 1$. Putting all of this together, if $x \in I_\delta(c)$, then there is some $r \in \mathbb{R}$ such that $|r| < \delta$ such that $x = c + r$. Then

$$\begin{aligned}
|p(x) - p(c)| &= |p(c+r) - p(c)| \\
&= \left|\sum_{i=1}^{n} a_i \sum_{k=1}^{i}\binom{i}{k} c^{i-k} r^k\right| \\
&\leq \sum_{i=1}^{n}|a_i|\sum_{k=1}^{i}\binom{i}{k}|c^{i-k}||r^k| \\
&< \sum_{i=1}^{n}|a_i|\sum_{k=1}^{i}\binom{i}{k}|c^{i-k}|\delta^k \\
&< \delta\sum_{i=1}^{n}|a_i|\sum_{k=1}^{i}\binom{i}{k}|c^{i-k}| \\
&\leq \varepsilon\,.
\end{aligned}$$

The second equality follows from Equation B.10. The first inequality follows from the Triangle Inequality 1.16. The last inequality follows from the definition of δ. Hence, $p(c) - p(x) < p(c)$. Hence, $p(x) > 0$. Since $x \in I_\delta(c)$ is arbitrary, it follows $p(x) \in I_\varepsilon(p(c))$. Hence, $p(I_\delta(c))$ is a subset of $I_\varepsilon(p(c))$. The lemma is proved. □

Theorem B.11. The Intermediate Value Theorem for Polynomials

Given a polynomial p and real numbers, x_1, x_2, if $p(x_1) < 0$ and $p(x_2) > 0$, then there is some c in between x_1 and x_2 such that $p(c) = 0$. That is, as a polynomial changes between positive and negative, it has a zero.

Proof. Suppose p is a polynomial, x_1, x_2 are real numbers, and $p(x_1) < 0$, $p(x_2) > 0$. Suppose $x_1 < x_2$. The other case, when $x_1 > x_2$, is similar. Define the set $S = \{x \in \mathbb{R} \mid x_1 \leq x < x_2 \text{ and } p(x) < 0\}$. This set is nonempty since $x_1 \in S$. Also, x_2 is an upper bound of S. Hence, by the least upper bound property, a least upper bound of S exists in $\mathbb{R}$, say c. We claim $p(c) = 0$.

First, we prove $p(c) \leq 0$. We do this by assuming $p(c) > 0$ and arriving at a contradiction. If $p(c) > 0$, by the above lemma, there is some $\delta > 0$ such that $p(x) > 0$ if $c - \delta < x \leq c$. (Choose $\varepsilon = p(c)$ in the above lemma to find such a δ.) Since c is the least upper bound of S, there exists some $x_0 \in S$ such that $c - \delta < x_0 \leq c$. In particular, $p(x_0) < 0$. But we can't have both $p(x_0) > 0$ and $p(x_0) < 0$. This is a contradiction. Hence, $p(c) \leq 0$.

Next, suppose $p(c) < 0$. Now recall $c \leq x_2$, since c is the least upper bound of S and x_2 is an upper bound of S. Hence, since $p(x_2) > 0$, and p is a function, $c < x_2$. By the above lemma, there is some $\delta > 0$ such that $p(x) < 0$ if $c \leq x < c + \delta$. (Choose $\varepsilon = -p(c)$ in the above lemma to find such a δ.) Since $c < x_2$, we can also choose δ so that $c + \frac{\delta}{2} < x_2$. Hence, in particular, $x_1 < c + \frac{\delta}{2} < x_2$ and $p(c + \frac{\delta}{2}) < 0$. That is, $c + \frac{\delta}{2} \in S$. But this contradicts that c is an upper bound of S, since $c + \frac{\delta}{2} > c$.

Hence, since $p(c)$ is neither positive nor negative, $p(c) = 0$ by Trichotomy. This also means, since $x_1 \leq c \leq x_2$ but $p(x_1) < 0$ and $p(x_2) > 0$, and that p is a function, that $x_1 < c < x_2$. That is, we've found some c in between x_1 and x_2 such that $p(c) = 0$.

In case $x_1 > x_2$, define the set $S = \{x \in \mathbb{R} \mid x_2 \leq x < x_1 \text{ and } p(x) > 0\}$ and proceed in the same way as above. □

As we'll note below, it follows every polynomial of odd degree has at least one real zero. This is because a polynomial of odd degree is eventually positive near one infinity and negative near the other, at some point the graph must cross the horizontal axis to go from positive to negative or vice versa. The guarantee of crossing follows from the intermediate value theorem.

Lemma B.12.

- If $a, b \in \mathbb{R}$, $a \leq b$, and p is a polynomial, then there is some $N \in \mathbb{R}$ for which $p(x) \leq N$ for all $x \in [a, b]$.
- If $a, b \in \mathbb{R}$, $a \leq b$, and p is a polynomial, then there is some $N \in \mathbb{R}$ for which $p(x) \geq N$ for all $x \in [a, b]$.

This is to say the image of any closed and bounded interval under any polynomial is bounded from above and below.

Proof.

- Suppose $a, b \in \mathbb{R}$, $a \leq b$ and p is a polynomial. Then there is some natural number n and real numbers $a_1, \ldots, a_n$ such that $p(x) = \sum_{i=0}^{n} a_i x^i$ for all real numbers x. Let c be either $|a|$ or $|b|$, whichever is greater than or equal to the other. Then, if $x \in [a, b]$, then $|x| \leq c$. Let $N = \sum_{i=0}^{n} |a_i| c^i$. Hence, for all $x \in [a, b]$,

$$p(x) \leq |p(x)| \leq \sum_{i=0}^{n} |a_i||x^i| = \sum_{i=0}^{n} |a_i||x|^i \leq \sum_{i=0}^{n} |a_i| c^i = N.$$

- Suppose $a, b \in \mathbb{R}$, $a \leq b$ and p is a polynomial. Then $-p$ is also a polynomial. Then, by the above item, there is some $-N \in \mathbb{R}$ for which $-p(x) \leq -N$ for all $x \in [a, b]$. Hence, $p(x) \geq N$ for all $x \in [a, b]$.

□

Lemma B.13. Every bounded sequence contains a convergent subsequence.

If f is a real function whose domain is the natural numbers, and whose range is contained in a closed and bounded interval $[a, b]$ for some real numbers $a \leq b$, then there exists an unbounded subset $S = \{m_n \in \mathbb{N} \mid n \in \mathbb{N}\}$ of the natural numbers and $y \in [a, b]$ such that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $|f(m_n) - y| < \varepsilon$.

Proof. Suppose f is a real function whose domain is the natural numbers, and whose range is contained in a bounded interval, say $[a, b]$ for some real numbers $a \leq b$. For every $n \in \mathbb{N}$, we define the set of real numbers $S_n = \{f(m) \in \mathbb{R} \mid m \geq n\}$. Then S_n is a subset of $[a, b]$, since it's a subset of range(f). Hence, for every $n \in \mathbb{N}$, S_n is bounded. It's also nonempty since $f(n) \in S_n$. Hence, the least upper bound of S_n exists and is a real number, say y_n. We now consider the set $\{-y_n \in \mathbb{R} \mid n \in \mathbb{N}\}$. This set is nonempty and is bounded from above by $-a$ because $a \leq f(n)$ for all n. Hence, the least upper bound of this set exists, say $-y \in \mathbb{R}$. Then, for every $N \in \mathbb{N}$, there exists $n_N \in \mathbb{N}$ such that $0 \leq -y + y_{n_N} \leq \frac{1}{2N}$. And because y_{n_N} is the least upper bound of S_{n_N}, there exists $m_{n_N} \geq n_N$ such that $0 \leq y_{n_N} - f(m_{n_N}) \leq \frac{1}{2N}$. Then, by the Triangle Inequality,

$$|f(m_{n_N})-y| = |f(m_{n_N})-y_{n_N}+y_{n_N}-y| \leq |f(m_{n_N})-y_{n_N}|+|y_{n_N}-y| = y_{n_N}-f(m_{n_N})+y_{n_N}-y \leq \frac{1}{2N}+\frac{1}{2N} = \frac{1}{N}.$$

Now, for every natural number n, define the natural number $m_n = m_{n_n}$. We define the set $S = \{m_n \in \mathbb{N} \mid n \in \mathbb{N}\}$. Then, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\varepsilon > \frac{1}{N}$. And if $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N}$. Hence, if $n \geq N$, then

$$|f(m_n) - y| \leq \frac{1}{N} < \varepsilon\,.$$

We show $y \in [a, b]$. $y \geq a$ since $-y_n \leq -a$ for all $n \in \mathbb{N}$. And $-y \geq -y_n \geq -b$ since b is an upper bound of S_n.

Now we argue the set $\{n_N \in \mathbb{N} \mid N \in \mathbb{N}\}$ is unbounded. Hence, S is unbounded. We note since, if $n \leq m$, then S_m is a subset of S_n, that $y_m \leq y_n$. Hence, since $y \leq y_n$ for all n, suppose there is some least n for which $y = y_n$. Then $y \leq y_m \leq y_n$ for all $m \geq n$. Hence, $y = y_m$ for all $m \geq n$. Hence, we may choose $\{n_N \in \mathbb{N} \mid N \in \mathbb{N}\}$ to be unbounded. Now suppose $y < y_n$ for all n. If there exists some $N, M \in \mathbb{N}$ such that $-y + y_n > \frac{1}{N}$ for all $n > M$, then we let ε be half of the least number out of the finite set $\{\frac{1}{N}\} \cup \{-y + y_i \in \mathbb{R} \mid i \in \{1, \ldots, M\}\}$. Then $\varepsilon > 0$ since $y - y_n < 0$ for all $n \in \mathbb{N}$ by assumption. And $y - y_n > \varepsilon$ for all $n \in \mathbb{N}$. This is a contradiction since y is the least upper bound of $\{y_n \mid n \in \mathbb{N}\}$. Hence, we conclude, for all $N, M \in \mathbb{N}$, there is some $n > M$ such that $-y + y_n < \frac{1}{N}$. We may let $M = N$ and $n = n_N$. Hence, $\{n_N \in \mathbb{N} \mid N \in \mathbb{N}\}$ is unbounded. Hence, $S\{m_n \in \mathbb{N} \mid n \in \mathbb{N}\}$ is unbounded since $m_{n_N} \geq n_N$ for all $N \in \mathbb{N}$. □

The above lemma is sometimes called the Bolzano–Weierstrass theorem, but if we refer to it, we'll refer to it as the sequential compactness characterization of the reals.

Theorem B.14. The Extreme Value Theorem for Polynomials

If $a, b \in \mathbb{R}$, $a \leq b$, and p is a polynomial, then there is some $y \in [a, b]$ for which $p(y) \geq p(x)$ for all $x \in [a, b]$. There is also some $w \in [a, b]$ such that $p(w) \leq p(x)$ for all $x \in [a, b]$. That is, for any polynomial p, for any real numbers a, b, the restricted function $p|_{[a,b]}$ has both a maximum and minimum value.

Proof. Suppose $a, b \in \mathbb{R}$, $a \leq b$ and p is a polynomial. Define the set $S = p([a, b])$, the image of $[a, b]$ under p. S is nonempty since $p(a) \in S$. And S is bounded above by the above lemma. Hence, S has a least upper bound, say z_0. The proof will conclude if we demonstrate $z_0 \in S$, for then there is some $y \in [a, b]$ such that $z_0 = p(y)$ and $p(y) \geq p(x)$ for all $x \in [a, b]$.

For every $n \in \mathbb{N}$, there exists some $x_n \in [a,b]$ for which $0 \leq z_0 - p(x_n) \leq \frac{1}{n}$. Now define the function f with domain $\mathbb{N}$ and codomain $[a,b]$ such that $f(n) = x_n$ if $n \in \mathbb{N}$. Then by the above lemma, there is some function g with domain and codomain $\mathbb{N}$, whose range is unbounded, such that there is some $y \in \mathbb{R}$ such that, for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $f \circ g(\{n \in \mathbb{N} \mid n \geq N\})$ is a subset of $I_\delta(y)$. Also, by the continuity of polynomials, Lemma B.9, for any $\varepsilon > 0$, there is some $\delta > 0$ such that $p(I_\delta(y))$ is a subset of $I_{\frac{\varepsilon}{2}}(f(y))$. Hence, if we choose $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$ and $f \circ g(\{n \in \mathbb{N} \mid n \geq N\})$ is a subset of $I_\delta(y)$, then by the Triangle Inequality,

$$|p(y)-z_0| = |p(y)-p(f(g(N)))+p(f(g(N)))-z_0| \leq |p(y)-p(f(g(N)))|+|p(f(g(N)))-z_0| < \frac{\varepsilon}{2}+\frac{1}{N} < \varepsilon\,.$$

Now, since $\varepsilon > 0$ is arbitrary, it follows $p(y) = z_0$. Since otherwise, $|p(y) - z_0| > 0$, so the above computation shows $|p(y) - z_0| < |p(y) - z_0|$, a contradiction. Hence, there exists some $y \in [a,b]$ such that $p(y) \geq p(x)$ for all $x \in [a,b]$. That is, $p|_{[a,b]}$ has a maximum value.

Since p is arbitrary, the above work also shows $-p$ has a maximum value on $[a,b]$. Hence, there exists some $w \in [a,b]$ such that $-p(w) \geq -p(x)$ for all $x \in [a,b]$. Hence, $p(w) \leq p(x)$ for all $x \in [a,b]$. Hemce, $p|_{[a,b]}$ has a minimum value. □

B.2 Exponentials and their continuity

Lemma B.15. If x is positive and y is a nonnegative real number, n is a natural number, and, for every $\varepsilon > 0$, $(y - \varepsilon)^n < x$, then $y^n \leq x$.

Proof. Suppose x is positive and y is a nonnegative real number and n is a natural number.

For any real ε, by the Binomial Theorem A.16,

$$(y - \varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} y^{n-k}(-\,\varepsilon)^k.$$

Now suppose $0 < \varepsilon \leq 1$. Then $\varepsilon^k \leq \varepsilon$ for any natural number k. Hence, if k is **odd** (not even), $(-\,\varepsilon)^k \geq -\,\varepsilon$ by Exercise 1.14. Also, if k is even, then $(-\,\varepsilon)^k > 0 > -\,\varepsilon$. Hence, since $y \geq 0$ and $\binom{n}{k} > 0$ for any natural numbers n and k for which $k \leq n$, it follows $\binom{n}{k} y^{n-k}(-\,\varepsilon)^k \geq \binom{n}{k} y^{n-k}(-\,\varepsilon)$. Hence,

$$\sum_{k=0}^{n} \binom{n}{k} y^{n-k}(-\,\varepsilon)^k = y^n + \sum_{k=1}^{n} \binom{n}{k} y^{n-k}(-\,\varepsilon)^k \geq y^n - \varepsilon \sum_{k=1}^{n} \binom{n}{k} y^{n-k}.$$

Notice this is true for every real ε for which $0 < \varepsilon \leq 1$.

Now we specialize and suppose $y^n > x$. We let ε be either $\frac{y^n - x}{2\sum_{k=1}^{n}\binom{n}{k} y^{n-k}}$, or 1, whichever is less than or equal to the other. Then $0 < \varepsilon \leq 1$, $-\,\varepsilon \geq -\frac{y^n - x}{2\sum_{k=1}^{n}\binom{n}{k} y^{n-k}}$, and

$$(y - \varepsilon)^n \geq y^n - \varepsilon \sum_{k=1}^{n} \binom{n}{k} y^{n-k} \geq y^n - \frac{y^n - x}{2} = \frac{y^n + x}{2} > x.$$

What we have demonstrated is if $y^n > x$, then there is some real number $\varepsilon > 0$ for which $(y - \varepsilon)^n \geq x$. This is, in fact, equivalent to the statement (and is the contrapositive of) if, for all real numbers ε, $\varepsilon > 0$ implies $(y - \varepsilon)^n < x$, then $y^n \leq x$. So, the lemma is proved. □

Theorem B.16. The existence and uniqueness of positive nth roots

If n is a natural number and x is a positive real number, there exists a unique positive number y for which $y^n = x$. We denote such a y by $x^{\frac{1}{n}}$ and call this number the nth root of x.

Proof. Let n be a natural number and x a positive real number. We consider the set S of all real numbers z for which $z^n < x$. Since $0^n = 0 < x$, it follows 0 is in S. Hence, S is nonempty. It is bounded because x is an upper bound of it. Hence, S has a least upper bound by the least upper bound property. Say y is the least upper bound of S. We claim $y^n = x$ and there is no other positive real number with this property (uniqueness). Since y is the least upper bound of S, by Exercise B.3, for every $\varepsilon > 0$, there is some z in S for which $y - \varepsilon \leq z$. Hence, by Exercise A.10, $(y-\varepsilon)^n \leq z^n$. Hence, since $z^n < x$ for every z in S, for every $\varepsilon > 0$, $(y-\varepsilon)^n < x$ by Trichotomy. This implies $y^n \leq x$ by Lemma B.15. That is, either $y^n < x$ or $y^n = x$.

We demonstrate $y^n < x$ leads to a contradiction. Again, by the Binomial Theorem, for any real ε for which $0 < \varepsilon < 1$, $\varepsilon^k < \varepsilon$ for all natural numbers $k > 1$,

$$(y+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \varepsilon^k < y^n + \varepsilon \sum_{k=1}^{n} \binom{n}{k} y^{n-k}.$$

Then we let ε be either $\frac{1}{2}$ or $\frac{x-y^n}{\sum_{k=1}^{n}\binom{n}{k}y^{n-k}}$, whichever one is less than or equal to the other. Then, if $y^n < x$, then $0 < \varepsilon < 1$ and

$$(y+\varepsilon)^n < y^n + \varepsilon \sum_{k=1}^{n} \binom{n}{k} y^{n-k} \leq y^n + \frac{x-y^n}{\sum_{k=1}^{n}\binom{n}{k}y^{n-k}} \cdot \sum_{k=1}^{n} \binom{n}{k} y^{n-k} = x.$$

Hence, $y+\varepsilon$ is in S. But $y < y + \varepsilon$ and y is an upper bound of S, a contradiction.

Hence, since $y^n \leq x$ and $y^n \not< x$, it follows $y^n = x$. This also demonstrates why y is positive. y is unique because any other number which is a positive nth root of x must also be the least upper bound of the set S. Since least upper bounds are unique, it follows y is unique. □

Theorem B.17.

(a) If n is an odd natural number and x is a negative real number, there is a unique negative number y for which $y^n = x$. Again we say y is the nth root of x and write $y = x^{\frac{1}{n}}$.

(b) If n is an odd natural number and x is a real number, then $(-x)^{\frac{1}{n}} = -(x^{\frac{1}{n}})$.

Proof. (a) If x is negative, then $-x$ is positive by Trichotomy and by Theorem B.16, the existence and uniqueness of positive nth roots, there is a unique positive number, $-y$ for which $(-y)^n = -x$. If n is odd, then there is some natural number m such that $n = 2m+1$. By Exercises 1.14 and A.10,

$$(-y)^n = (-y)^{2m+1} = (-y)(-y)^{2m} = -y \cdot ((-y)^m)^2 = -y \cdot (y^m)^2 = -y \cdot y^{2m} = -(y^{2m+1}) = -y^n.$$

This means

$$-y^n = -x.$$

Hence,

$$y^n = x.$$

Now, y is negative since $-y$ is positive. This proves existence.

For uniqueness, suppose there is some z for which $z^n = x$. Then $(-z)^n = -x$ by the above computation. Hence, $-z$ is positive, so that $-z = -y$ by uniqueness of positive nth roots. Hence, $z = y$. This proves uniqueness.

(b) Notice the above computation shows $-(-y)^n = x$. Since $y^n = x$, $-y = (-x)^{\frac{1}{n}}$, and $y = x^{\frac{1}{n}}$ by definition, the result $(-x)^{\frac{1}{n}} = -(x^{\frac{1}{n}})$ follows from uniqueness.

□

Exercise B.18. The integer case analogous to Exercise A.10

If a, b are real numbers and n, m are nonzero integers,

(a) Prove, if $a = b$, then $a^n = b^n$. Hint: one case: when $n < 0$.

(b) Prove, if $m = n$, then $a^m = a^n$. Hint: one case: when $m < 0$.

(c) Prove $(ab)^n = a^n b^n$. Hint: one case: when $n < 0$.

(d) Prove $a^{m+n} = a^m \cdot a^n$. Hint: three cases: when $m, n < 0$, when $m+n < 0$, $m < 0$, $n > 0$, when $m + n > 0$, $m < 0$, $n > 0$.

(e) Prove $(a^m)^n = a^{mn}$. Hint: two cases: when $m, n < 0$, when $m < 0$, $n > 0$.

(f) Prove, if $0 < a < b$ and $n < 0$, then $b^n < a^n$. Deduce, if $a > 0$, $b > 0$, $n < 0$ and $b^n < a^n$, then $a < b$.

(g) Prove, if $1 < a$ and $m < n$, then $a^m < a^n$. Deduce, if $a > 1$ and $a^m < a^n$, then $m < n$.

Exercise B.19. The rational case analogous to Exercise B.18

If a, b are positive numbers, m, n, p, q are integers and n and p are nonzero,

(a) Prove, if $a = b$, then $a^{\frac{m}{n}} = b^{\frac{m}{n}}$.

(b) Prove, if $\frac{m}{n} = \frac{p}{q}$, then $a^{\frac{m}{n}} = a^{\frac{p}{q}}$.

(c) Prove $(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} b^{\frac{m}{n}}$.

(d) Prove $a^{\frac{m}{n}+\frac{p}{q}} = a^{\frac{m}{n}} \cdot a^{\frac{p}{q}}$.

(e) Prove $(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{mp}{nq}}$.

(f) Prove, if $a < b$ and $n > 0$, then $a^{\frac{1}{n}} < b^{\frac{1}{n}}$. Deduce, if $n > 0$ and $a^{\frac{1}{n}} < b^{\frac{1}{n}}$, then $a < b$.

(g) Prove, if $1 < a$ and $\frac{m}{n} < \frac{p}{q}$, then $a^{\frac{m}{n}} < a^{\frac{p}{q}}$. Deduce, if $a > 1$ and $a^{\frac{m}{n}} < a^{\frac{p}{q}}$, then $\frac{m}{n} < \frac{p}{q}$.

Lemma B.20. If m is an integer, n and p are nonzero integers, and x is a positive number, then

$$(x^m)^{\frac{1}{n}} = (x^{pm})^{\frac{1}{pn}}.$$

Proof. Suppose m is an integer, n and p are nonzero integers, and x is a positive number. From Exercises B.19 e and b, it follows

$$(x^{pm})^{\frac{1}{pn}} = x^{\frac{pm}{pn}} = (x^{\frac{m}{n}})^{\frac{p}{p}} = (x^{\frac{m}{n}})^1 = x^{\frac{m}{n}}.$$

□

Definition B.21.

(a) If q is a rational number, and x is a positive real number, the above lemma means we can define x^q without ambiguity. There is an integer m and a natural number n for which $q = \frac{m}{n}$. Then we define x^q by $x^{\frac{m}{n}}$.

(b) Finally, if a is any real number, b is a positive real number and $b \geq 1$, we define **the ath power of b** by the least upper bound of the set

$$\{z \in \mathbb{R} \mid z = b^q \text{ for some } q \in \mathbb{Q} \text{ with } q \leq a\}$$

and denote this number by b^a. We say b^a is a **power** of b, a is the **exponent** of b^a, and b is the **base** of b^a. Below we demonstrate if $b > 0$, $b \geq 1$ and $a \in \mathbb{R}$, then $b^a \in \mathbb{R}$. In fact, $b^a > 0$.

- If $b > 0$, $b \geq 1$ and $a \in \mathbb{R}$, let $S = \{z \in \mathbb{R} \mid z = b^q \text{ for some } q \in \mathbb{Q} \text{ with } q \leq a\}$. We need to show S is bounded and nonempty.
  - Find a $q \in \mathbb{Q}$ and $q \leq a$ by the Density of the rationals. Then $b^q \in S$. Hence, S is nonempty.
  - Again by the density of $\mathbb{Q}$, find a rational $q \geq a$. Then by Exercises B.19 b and g, for every $p \leq q$, if $p \in \mathbb{Q}$, $b^p \leq b^q$. Hence, b^q is an upper bound of S.
- Since S is nonempty and bounded from above, it follows its least upper bound exists by the least upper bound property. In fact, since $b^q > 0$ for any $q \in \mathbb{Q}$, it follows $b^a > 0$ as well.

(c) If a is a real number, b is a real number, and $0 < b < 1$, we define b^a by $(b^{-a})^{-1}$.

■

Exercise B.22. The real case analogous to Exercise B.19.

If b and c are positive, nonone real numbers, a and r are real numbers, explain the following statements.

(a) If $b = c$, then $b^a = c^a$.

(b) If $a = r$, then $b^a = b^r$.

(c) $(bc)^a = b^a c^a$.

(d) $b^{a+r} = b^a \cdot b^r$.

(e) $(b^a)^r = b^{(ar)}$.

(f) If $b < c$ and $a > 0$, then $b^a < c^a$.

(g) If $a < r$ and $b > 1$, then $b^a < b^r$.

Exercise B.23.

(a) If $a \geq 1$, then either $a = 1$ or, for all $M > 0$, there exists some $n \in \mathbb{N}$ such that $a^n > M$. *Hint:* Suppose there is some $M > 0$ for which $a^n \leq M$ for all $n \in \mathbb{N}$. Then consider the least upper bound of the the set $\{a^n \mid n \in \mathbb{N}\}$ which exists by our supposition, since M is an upper bound of this set. Label this least upper bound b. Then, for all $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that $b - a^n < \varepsilon$. By Exercise A.10, it follows $0 \leq a^{n+1} - a^n$. Conclude $a^{n+1} - a^n < \varepsilon$. Hence, conclude, for all $\varepsilon > 0$, $0 \leq a - 1 < \varepsilon$. This implies $a = 1$. Hence, if $a > 1$, then, for all $M > 0$, there exists some $n \in \mathbb{N}$ such that $a^n > M$.

(b) If $0 < a < 1$, then for all $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that $a^n < \varepsilon$.

(c) If $b > 1$, then for all $\varepsilon > 0$, there exists some $q \in \mathbb{Q}$ such that $q < 0$ and $1 - b^q < \varepsilon$.

Prove, for any $b > 0$, for any $x \in \mathbb{R}$, b^x is the least upper bound of the set $\{b^q \mid q \in \mathbb{Q} \text{ and } q < x\}$. Do so by demonstrating for every $\varepsilon > 0$, there is some $q \in \mathbb{Q}$ such that $q < x$ and $b^x - b^q < \varepsilon$.

Lemma B.24. Continuity of the exponential at 0

If a is a positive, nonone real number, then, for all $\varepsilon \in \mathbb{R}$, if $\varepsilon > 0$, then there exists some $\delta \in \mathbb{R}$ such that $\delta > 0$ and the image $\exp_a(I_\delta(0))$ is a subset of the open interval $I_\varepsilon(1)$.

Proof. Suppose a is a positive, nonone real number and ε is a positive real number. Suppose first $a > 1$. By Exercise B.23, there is some $0 < \delta \in \mathbb{Q}$ such that $0 \leq 1 - a^{-\delta} < \frac{\varepsilon}{a}$. Since $\exp_a$ is increasing on $\mathbb{R}$ by Exercise B.22 g, it follows, if $0 \leq x < \delta$, then $-x > -\delta$, so, $a^{-x} > a^{-\delta}$. Hence,

$$\text{if } -\delta < -x \leq 0, \text{ then } 0 \leq 1 - a^{-x} < 1 - a^{-\delta} < \frac{\varepsilon}{a}. \tag{B.25}$$

Now, since $a > 1$, by Exercise 1.14 g, $\frac{1}{a} < 1$. Hence,

$$\text{if } -\delta < x \leq 0, \text{ then } |a^x - 1| < \varepsilon. \tag{B.26}$$

Moreover, we may assume $\delta < 1$, since otherwise we could choose $\frac{1}{2}$ and our previous statements would still hold, again by the increasingness of $\exp_a$. Since $\delta < 1$, then if $0 < x < \delta$, then $0 < a^x < a$, again since $\exp_a$ is increasing. If $0 < x < \delta$, then $-\delta < -x < 0$. Hence, by Statement B.25 and Exercise B.22, if $0 < x < \delta$, then

$$|a^x - 1| = a^x - 1 = a^x(1 - a^{-x}) < a^x \cdot \frac{\varepsilon}{a} < \varepsilon.$$

$$\text{if } 0 < x < \delta \text{ then } |a^x - 1| < \varepsilon. \tag{B.27}$$

By Exercise B.8, $-\delta < x < \delta$ if and only if $x \in I_\delta(0)$ and $|a^x - 1| < \varepsilon$ if and only if $\exp_a(x) \in I_\varepsilon(1)$. Hence, combining Statements B.26 and B.27, it follows, if $x \in I_\delta(0)$, then $\exp_a(x) \in I_\varepsilon(1)$. Hence, $\exp_a(I_\delta(0))$ is a subset of the open interval $I_\varepsilon(1)$.

This was the case $a > 1$. Now, if $0 < a < 1$, then by definition, for all $x \in \mathbb{R}$, $a^x = (a^{-1})^{-x}$. Again, $a^{-1} > 1$ if $0 < a < 1$. Hence, by the above argument, there is some $\delta > 0$ such that $|x| < \delta$ implies $|(a^{-1})^x - 1| < \varepsilon$. But $|-x| = |x|$, so, if $|x| < \delta$, then $|(a^{-1})^{-x} - 1| < \varepsilon$. Hence, $|a^x - 1| < \varepsilon$. Thus, the case when $0 < a < 1$ is proved as well. □

Since $\exp_a(x+y) = \exp_a(x)\exp_a(y)$, for all real x, y, for all positive, nonone a, the behavior of exp anywhere is determined by its behavior at zero.

Lemma B.28. Continuity of Exponentials

If a is a positive, nonone real number and c is a real number, then, for all $\varepsilon \in \mathbb{R}$, if $\varepsilon > 0$, then there exists some $\delta \in \mathbb{R}$ such that $\delta > 0$ and the image $\exp_a(I_\delta(c))$ is a subset of the open interval $I_\varepsilon(\exp_a(c))$.

Proof. Suppose a is a positive, nonone real number, c is a real number, and ε is a positive real number. By Lemma B.24, there is some $\delta > 0$ such that $|x| < \delta$ implies $|a^x - 1| < \frac{\varepsilon}{a^c}$. If $|x - c| < \delta$, then

$$|a^x - a^c| = a^c|a^{x-c} - 1| < a^c \cdot \frac{\varepsilon}{a^c} = \varepsilon\,.$$

Notice $a^x - a^c = a^c(a^{x-c} - 1)$ by various parts of Exercise B.22. And recall $a^c > 0$ by Exercise B.22 g, so that $|a^x - a^c| = |a^c(a^{x-c} - 1)| = a^c|a^{x-c} - 1|$ by Exercise 1.17. □